Higher Contou-Carrére symbol

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Preliminaries

Let A be a commutative associative ring with a unit. Let $A((t))$ be a ring of Laurent series in one variable.

$$A((t)) = \left\{ \sum_{i \geq m} a_i t^i | a_i \in A \right\}.$$

It has a canonical topology, which is given by a sequence of neighborhoods of zero $t^k A[[t]]$.

Structure of multiplicative group

For simplicity suppose that $A \neq A_1 \times A_2$ or equivalently $\mathrm{Spec}\, A$ is connected.

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Every element $f \in A((t))^*$ has a unique decomposition:

$$f = a_0 t^{v(f)} \prod_{i \geq 1} (1 - a_{-i} t^{-i}) \prod_{i \geq 1} (1 - a_i t^i),$$

where $a_0 \in A^*, v(f) \in \mathbb{Z}, a_i \in A, a_{-i} \in \mathrm{Nil}(A)$ and $a_{-i} = 0$ for $i >> 0$.

Contou-Carrére symbol

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$$CC_1 : A((t))^* \times A((t))^* \to A^*.$$

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It can be given by an explicit formula. Given $f, g \in A((t))^*$ we have the following decomposition:

$$f = a_0 t^{v(f)} \prod_{i \geq 1} (1 - a_{-i} t^{-i}) \prod_{i \geq 1} (1 - a_i t^i),$$
$$g = b_0 t^{v(g)} \prod_{i \geq 1} (1 - b_{-i} t^{-i}) \prod_{i \geq 1} (1 - b_i t^i).$$

Contou-Carrére symbol is given by the formula:

$$CC_1(f, g) = (-1)^{v(f)v(g)} \frac{a_0^{v(g)} \prod_{i,j>0} (1 - a_i^{\frac{j}{(i,j)}} b_{-j}^{\frac{j}{(i,j)}})}{b_0^{v(f)} \prod_{i,j>0} (1 - b_i^{\frac{j}{(i,j)}} a_{-j}^{\frac{j}{(i,j)}})}.$$

Deligne's formula

There is a simple formula for CC_1 which is due to Deligne.

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We say that an element $f \in A((t))$ is *topologically nilpotent* if f^n converges to 0.
If A is an $\mathbb{Q}-$algebra then for every topologically nilpotent element $f \in A((t))$ and an invertible element $g \in A((t))^*$ we have

$$CC_1(1+f,g) = \exp \mathrm{Res}(\log(1+f)\frac{dg}{g})),$$

where the $\log(1+f)$ is given by standard series:

$$\log(1+f) = \sum_{i=0}^{\infty}(-1)^{i+1}\frac{f^i}{i}.$$

Reciprocity law

In 2001 A. Beilinson, S. Bloch, H. Esnault and independently G. Anderson, F. Pablos Romo in 2002 gave interpretation of Contou-Carrére symbol in terms of *determinant groupoids*. In their works Contou-Carrére symbol appears as the commutator of lifts of elements in a central extension of the group $A((t))^*$ by A^*. They proved the reciprocity law.

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Let C be a smooth projective curve over an algebraically closed field F.
Then for every closed point $x \in C$ we define the *completed* local ring:

$$\hat{\mathcal{O}}_{C,x} = \varprojlim_{n \geq 1} \mathcal{O}_{C,x}/\mathfrak{m}^n.$$

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Given local parameter $t_x \in \mathcal{O}_{C,x}$, there is an isomorphism $\hat{\mathcal{O}}_{C,x} \cong F[[t_x]]$. For a finite artinian F-algebra A and an element f of $(F(C) \otimes A)^*$ we define element f_x as the image under the map:

$$(F(C) \otimes A)^* \to (\mathrm{Quot}(\hat{\mathcal{O}}_{C,x}) \otimes A)^* = A((t))^*.$$

Reciprocity law

Theorem (A. Beilinson, S. Bloch, H. Esnault and G. Anderson, F. Pablos Romo)

Let C be a smooth projective curve over an algebraically closed field F, A be a finite artinian F-algebra. Then for every $f, g \in (F(C) \otimes A)^*$ one have:

$$\prod_{x \in C} CC_1(f_x, g_x) = 1,$$

where the product is taken over all closed points of the curve C.

Reciprocity law

Example

Let A be a field. Then CC_1 coincides with the tame symbol. Then the reciprocity law gives the well-known Weil reciprocity law.

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Let $A = k[\varepsilon]$, where $\varepsilon^3 = 0$ and k is a field. Then for every $f, g \in k((t))$ one can compute: $CC_1(1+\varepsilon f, 1+\varepsilon g) = 1 + \varepsilon^2 \mathrm{Res}(fdg)$. Thus the reciprocity law for the Contou-Carrére symbol gives that the sum of residues of a differential form over all points of a projective smooth curve is zero.

Higher-dimensional case

- D. Osipov and X. Zhu defined the two dimensional Contou-Carrére symbol by giving 2-categorical generalization of determinant groupoids and also using algebraic K-theory. Concretely, they defined the functorial map

$$CC_2 : A((t))((s))^* \times A((t))((s))^* \times A((t))((s))^* \longrightarrow A^*,$$

which is polymultiplicative and antisymmetric. They also proved the reciprocity laws on smooth projective surfaces.

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  which is polymultiplicative and antisymmetric. They also proved the reciprocity laws on smooth projective surfaces.
- S. Gorchinskiy and D. Osipov for an arbitrary n defined the higher dimensional Contou-Carrére symbol using boundary maps in algebraic K-theory. Explicitly, they defined a functorial map

$$CC_n : A((t_1))...((t_n))^* \times ... \times A((t_1))...((t_n))^* \longrightarrow A^*,$$

  where we have $n+1$ copies of iterated Laurent series on the left. The symbol is antisymmetric and polymultiplicative. They proved an explicit formula which is similar to the one given by P. Deligne.

Universal property for CC_n

They proved that CC_n satisfies the Steinberg property, i.e. for every $f_0, ..., f_n \in A((t_1))...((t_n))^*$ such that $f_{i+1} = 1 - f_i$ for some i:

$$CC_n(f_0, ..., f_n) = 1.$$

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Theorem (S. Gorchinskiy, D. Osipov)

Suppose that for every ring A we have polymultiplicative functorial map

$$(\cdot, ..., \cdot) : A((t_1))...((t_n))^* \times ... \times A((t_1))...((t_n))^* \longrightarrow A^*,$$

which satysfies the Steinberg property. Then for some $m \in \mathbb{Z}$ we have

$$(f_0, ..., f_n) = CC_n(f_0, ..., f_n)^m.$$

Invariance under continuous automorphisms

The ring $A((t_1))...((t_n))$ has a topology which comes from the definition of this ring as an iterated ind-pro limit.

Example

If n $= 1$ then all continuous automorphisms of the ring $A((t))$ are in one to one correspondance with elements $f \in A((t))^*$ such that $v(f) = 1$.
Explicitly, if $\varphi : A((t)) \to A((t))$ is automorphism then it corresponds to an element $\varphi(t)$.

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Theorem (S. Gorchinskiy, D. Osipov)

If $\varphi : A((t_1))...((t_n)) \to A((t_1))...((t_n))$ is continuous ring automorphism then

$$CC_n(\varphi(f_0), ..., \varphi(f_n)) = CC_n(f_0, ..., f_n).$$

The "converse" also holds. In some sense CC_n can be characterized by this invariance property up to a sign.

Theorem

Suppose that for all rings A we have a polymultiplicative functorial map

$$(\cdot, ..., \cdot) : A((t_1))...((t_n))^* \times ... \times A((t_1))...((t_n))^* \longrightarrow A^*,$$

which is invariant under continuous automorpshisms, i.e. for every $\varphi : A((t_1))...((t_n)) \to A((t_1))...((t_n))$ *continuous automorphism*

$$(\varphi(f_0), ..., \varphi(f_n)) = (f_0, ..., f_n).$$

Then there are $m \in \mathbb{Z}$ *and*

$$\omega : A((t_1))...((t_n))^* \times ... \times A((t_1))...((t_n))^* \longrightarrow \{\pm 1\},$$

such that

$$(f_0, ..., f_n) = \omega(f_0, ..., f_n) CC_n(f_0, ..., f_n)^m.$$