Geometry and arithmetic of weighted complete intersections

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Geometric characterisation of arithmetic varieties (Belyi-type theorems)

Belyi's theorem and its generalisations

Theorem (Belyi, 1980)

A smooth complex projective curve X can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a morphism $X \to \mathbb{P}^1$ étale over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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Definition (González-Diez)

Let X be a smooth projective variety. A *Lefschetz function* is a composition of the rational map $X \dashrightarrow \mathbb{P}^1$ defined by some Lefschetz pencil on X with a rational function on $\mathbb{P}^1$.

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Theorem (González-Diez, 2008)

A smooth complex projective surface X of general type can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a Lefschetz function on X with at most 3 critical points.

Belyi's theorem for complete intersections of general type

Theorem (Javanpeykar, 2017)

Let $X \subset \mathbb{P}^n$ be a smooth complex complete intersection of general type of dimension at least 3. Then the variety X can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a Lefschetz function $X \dashrightarrow \mathbb{P}^1$ with at most 3 critical points.

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Theorem (—, 2024)

Let $X \subset Y$ be a smooth complex (well-formed) complete intersection of general type of dimension at least 3, where Y is either a generalised Grassmannian or a weighted projective space.

Then the variety X can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a Lefschetz function $X \dashrightarrow \mathbb{P}^1$ with at most 3 critical points.

Belyi's theorem for admissible complete intersections

Definition

Let Y be a complex Mori dream space with $\mathrm{Cl}(Y) \simeq \mathbb{Z}$ which can be defined over $\mathbb{Q}$. A smooth complete intersection $X \subseteq Y$ of multidegree $(d_1, \ldots, d_c)$ is *admissible* if

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- res: $H^0(Y, \mathcal{H}^{\otimes k}) \twoheadrightarrow H^0(X, \mathcal{H}|_X^{\otimes k})$ for any $k \in \mathbb{Z}_{\geqslant 0}$, and for any $D \in H^0(X, \mathcal{H}|_X^{\otimes k})$ there exists $\widetilde{X} \in H^0(Y, \mathcal{H}^{\otimes k})$ such that $X = \widetilde{X} \cap Y$ is an admissible complete intersection in Y;

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- for any admissible complete intersection $X \subset Y$ and any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ its conjugate X^σ is also admissible;
- ...

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Definition (continued)

- any smooth divisor $D \in H^0(X, \mathcal{H}|_X^{\otimes k})$ of general type in a very ample linear system satisfies the *infinitesimal Torelli theorem*:

$$H^1(X, \mathcal{T}_X) \hookrightarrow \bigoplus_{p+q=\dim(X)} \mathrm{Hom}(H^p(X, \Omega_X^q), H^{p+1}(X, \Omega_X^{q-1})).$$

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Theorem (—, 2024)

Let $X \subset Y$ be a smooth complex admissible complete intersection of general type of dimension at least 3. Then the variety X can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a Lefschetz function $X \dashrightarrow \mathbb{P}^1$ with at most 3 critical points.

Towards complete intersections in weighted generalised Grassmannians

Weighted generalised Grassmannians

Definition

A Mori dream space Y with $\mathrm{Cl}(Y) \simeq \mathbb{Z}$ is a *weighted generalised Grassmannian* if its Cox ring $\mathcal{R}(Y)$ admits homogeneous generators such that the associated ideal of relations coincides with the ideal of relations of the Cox ring of a usual generalised Grassmannian.

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Construction (Grojnowski, Corti–Reid)

A generalised Grassmannian $Y = G/P$ admits a minimal G-equivariant embedding $Y \hookrightarrow \mathbb{P}(W)$, where $\Psi\colon G \hookrightarrow \mathrm{GL}(W)$ is the corresponding fundamental representation. One can consider its affine cone $\widehat{Y} \subset W$. It admits a natural action of both G, and the centre $Z(\mathrm{GL}(W)) \simeq \mathbb{G}_m$.

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Explicit description of weighted Grassmannians

Remark

Any weighted projective space $\mathbb{P}(a_0, \ldots, a_N)$ that can be obtained in this way automatically satisfies $\sum a_i = 0$ modulo $N + 1$.

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Any weighted (classical) Grassmannian Y is isomorphic to a GIT quotient $\mathrm{Gr}_k(a_0, \ldots, a_n)$ by a positive coweight $\sum_{i=0}^{n} a_i \varphi_i^\vee$, where

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$$\varphi_i^\vee = \begin{cases} (k/(n+1) - 1) \cdot \tau_W^\vee + \Psi(\eta_i^\vee) & \text{for } 0 \leqslant i \leqslant k-2; \\ (k/(n+1)) \cdot \tau_W^\vee + \Psi(\eta_i^\vee) & \text{for } k-1 \leqslant i \leqslant n. \end{cases}$$

We put $\eta_i^\vee = -\omega_{i-1}^\vee + \omega_i^\vee$ for all $i = 0, \dots, n$, where $\omega_0^\vee = \omega_{n+1}^\vee = 0$, and $\omega_i^\vee$ are fundamental coweights of G.

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The coweight lattice can be realised an integral submodule of $\mathbb{Q}\langle\Psi(\eta_0^\vee),\ldots,\Psi(\eta_n^\vee),\deg(W)^{-1}\cdot\tau_W^\vee\rangle$ generated by columns of

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$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ \alpha & \alpha & \alpha & \cdots & \beta & \beta & \beta \end{pmatrix}, \begin{cases} \alpha = (\frac{k}{n+1} - 1)\binom{n+1}{k}, \\ \beta = \frac{k}{n+1}\binom{n+1}{k} = \binom{n}{k-1}, \\ \det = (n-k+2)\beta + \\ +(k-1)\alpha = \binom{n+1}{k}. \end{cases}$$

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The Weyl group acts by (inverse) permutations of $\Psi(\eta_i^\vee)$ (so it permutes the columns while preserving the last row). It is not hard to prove that the coweight lattice is a permutation representation of the Weyl group if and only if we have $\min(k, n+1-k) = 1$.

Explicit description of weighted Grassmannians

Corollary (—, 2024)

Any possible $\mathbb{Z}$-grading on Plücker coordinates is of the form

$$\deg(T_{i_1<\cdots<i_k}) = -\sum_{l=0}^{k-2} a_l + \sum_{j=1}^{k} a_{i_j}.$$

If Y is well-formed, then its dualising sheaf is isomorphic to

$$\omega_Y \simeq \mathcal{O}_Y\left(-k\left(\sum_{i=0}^{n} a_i\right) + (n+1)\left(\sum_{i=0}^{k-2} a_i\right)\right).$$

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Example

For $k=2$, $n=3$, the single Plücker relation is given by

$$T_{0,1}T_{2,3} + T_{1,2}T_{0,3} - T_{0,2}T_{1,3} = 0.$$

We can clearly see that it quasi-homogeneous of the degree

$$-a_0 + a_1 + a_2 + a_3 = \deg(-K_{\mathbb{P}}) - \deg(-K_Y),$$

which is just the adjunction formula for a weighted hypersurface.

Hilbert series of a weighted Grassmannian

Proposition (—, 2024)

Let $Y = \mathrm{Gr}_k(a_0, \dots, a_n)$ be a well-formed weighted Grassmannian. Its Hilbert series equals $\mathcal{H}(Y) = \mathcal{P}(Y)/\mathcal{Q}(Y)$, where

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$$\mathcal{P}(Y) = \sum_{I=(i_1<\dots<i_k)} \frac{F(I)}{1-t^{a(I)}}, \quad \mathcal{Q}(Y) = \prod_{\substack{i,j=0,\dots,n \\ i<j}} (1 - t^{a_j - a_i});$$

$$a(I) = -\sum_{l=0}^{k-2} a_l + \sum_{j=1}^{k} a_{i_j} > 0, \quad F(I) = \left(\sum_{\sigma \in S_I} (-1)^\sigma t^{f(\sigma)} \right);$$

$$S_I = \{\sigma \in S_{n+1} : \sigma(\{0, \dots, k-1\}) = I\},$$

$$f(\sigma) = \sum_{i=0}^{n} (i - \sigma^{-1}(i)) a_i.$$

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Remark

It seems that there exists a unified approach to the proof of infinitesimal Torelli theorem in terms of Jacobi rings for complete intersections in generalised Grassmannians (Fatighenti–Mongardi) and weighted projective spaces (Licht).

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Weighted generalised Grassmannians are Mori dream spaces with $\mathrm{Cl}(Y) \simeq \mathbb{Z}$ which can be covered by standard affine charts $\mathbb{A}^{\mathrm{dim}(Y)}/\mathcal{G}$, where $\mathcal{G}$ is a finite abelian group acting on the Plücker coordinates by multiplication on roots of unity.

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Problem

Find a structural classification of smooth well-formed Fano complete intersections in weighted generalised Grassmannians, similar to the classification of smooth Fano WCI (–).

Thank you!