Rigidity of Schubert classes in partial flag varieties

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Introduction

- A Schubert class σ is called rigid if it can only be represented by Schubert varieties;
- A Schubert class σ is called multi-rigid if $m\sigma$, $m \in \mathbb{Z}^+$, can only be represented by unions of Schubert varieties.

Problem (Borel and Haefliger, 1961)

Which Schubert classes can be represented by a smooth subvariety?

- If a Schubert class is rigid and singular, then it cannot be represented by a smooth subvariety;
- If a Schubert class is multi-rigid and singular, then any multiple of it cannot be represented by a positive linear combination of classes of smooth subvarieties.

Schubert varieties in $G(k, n)$

- V = complex vector space of dimension n.

$$\begin{array}{rcl} F_\bullet & : & F_{a_1} \subset F_{a_2} \subset \ldots \subset F_{a_k} \\ \dim(F_\bullet \cap \Lambda) & : & 1 \quad\quad 2 \quad\quad \ldots \quad\quad k \end{array}$$

$$\Sigma_{a_1,\ldots,a_k}(F_\bullet) := \{\Lambda \in G(k,n) | \dim(F_{a_i} \cap \Lambda) \geq i\}.$$

Remark. It is customary to parameterize Schubert classes in $G(k, n)$ by partitions $\lambda : n - k \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0$. The relation between the two notations is given by

$$a_i = n - k + i - \lambda_i$$

The rigidity problem in Grassmannians

Theorem (Coskun 2011)

Let σ_a be a Schubert class for $G(k, n)$. Set $a_0 = 0$ and $a_{k+1} = \infty$. The Schubert class σ_a is rigid if and only if for all sub-indices a_i, one of the following holds:

1. $a_i = a_{i+1} - 1$;
2. $a_i = a_{i-1} + 1$;
3. $a_i \leq a_{i+1} - 3$.

If $a_i = a_{i+1} - 1$, then $\dim(F_{a_{i+1}} \cap \Lambda) \geq i + 1 \Rightarrow \dim(F_{a_i} \cap \Lambda) \geq i$.

Definition

A sub-index a_i is called *essential* if $a_i \neq a_{i+1} - 1$.

Theorem

Let σ_a be a Schubert class for $G(k, n)$. Set $a_0 = 0$ and $a_{k+1} = \infty$. If either $a_i = a_{i-1} + 1$ or $a_i \leq a_{i+1} - 3$, then for every representative X of σ_a there exists a subspace F_{a_i} such that

$$\dim(F_{a_i} \cap \Lambda) \geq i, \forall \Lambda \in X$$

- A sub-index satisfying the conditions in the theorem is called *rigid*.

Partial flag varieties

- $F(d_1, ..., d_k; n) := \{\Lambda_\bullet = (\Lambda_1, ..., \Lambda_k) | \Lambda_t \in G(d_t, n), 1 \leq t \leq k\}$

$$F_\bullet^{\alpha_\bullet} : F_{a_1}^{\alpha_1} \subset F_{a_2}^{\alpha_2} \subset ... \subset F_{a_{d_k}}^{\alpha_{d_k}}$$

$$\Downarrow$$

$$F_\bullet^{\leq t}, 1 \leq t \leq k$$

$$\Downarrow$$

$$\Sigma(F_\bullet^{\leq t}) \subset G(d_t, n)$$

$$\Sigma_{a^\alpha}(F_\bullet) := \{\Lambda_\bullet \in F(d_1, ..., d_k; n) | \dim(F_{a_i} \cap \Lambda_t) \geq \mu_{i,t}\}$$

$$\mu_{i,t} := \{s | s \leq i, \alpha_s \leq t\}$$

- A sub-index a_i is called essential if it is essential with respect to $(\pi_t)_*(\sigma_{a^\alpha})$ for some $1 \leq t \leq k$, or equivalently either $a_i \neq a_{i+1} - 1$ or $\alpha_i < \alpha_{i+1}$.
- An essential sub-index a_i is called rigid if for every representative X of σ_{a^α}, there exists a subspace F_{a_i} of dimension a_i such that

$$\dim(F_{a_i} \cap \Lambda_j) \geq \mu_{i,j}, 1 \leq j \leq k, \forall \Lambda_\bullet \in X$$

Theorem

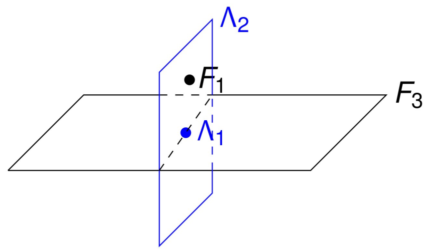
If a_i is essential with respect to $(\pi_t)_*(\sigma_{a^\alpha})$ and there exists a linear subspace F_{a_i} such that $\dim(F_{a_i} \cap \Lambda_t) \geq \mu_{i,t}, \forall \Lambda_\bullet \in X$, then $\dim(F_{a_i} \cap \Lambda_j) \geq \mu_{i,j}$ for all $1 \leq j \leq k$.

Corollary

An essential sub-index is rigid if and only if it is rigid with respect to $(\pi_t)_*(\sigma_{a^\alpha})$ for some $1 \leq t \leq k$.

Remark: A Schubert class may not be rigid even if all essential sub-indices are rigid.

$$\begin{array}{ccc} & \sigma_{1^2,3^1,4^2} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \sigma_3 & & \sigma_{1,3,4} \end{array}$$



$$X := \{(\Lambda_1, \Lambda_2) | \Lambda_1 \subset F_3, \Lambda_2 \supset F_1\} \subset F(1,3;4)$$

Lemma

Let σ_a be a Schubert class for $G(k,n)$. Let X be a representative of σ_a. Assume there are two linear subspaces F_{a_i} and F_{a_j} of dimension a_i and a_j respectively, such that $\dim(\Lambda \cap F_{a_i}) \geq i$, $\dim(\Lambda \cap F_{a_j}) \geq j$ for all $\Lambda \in X$. If $i < j$ and a_j is essential, then $F_{a_i} \subset F_{a_j}$.

We define a relation '$\to$' between two sub-indices:

- $a_i \to a_j$ if $i < j$ and a_j is essential in $(\pi_t)_*(\sigma_{a^\alpha})$ for some $t \geq \min(\alpha_i, \alpha_j)$.

This relation extends to a strict partial order (which we also denote by '$\to$') on the set of essential sub-indices by transitivity.

Corollary

A Schubert class $\sigma_{a^\alpha} \in A(F(d_1, ..., d_k; n))$ is rigid if and only if all essential sub-indices are rigid and the set of all essential sub-indices is strict totally ordered under the relation '$\to$'.

The multi-rigidity problem

Theorem (Hong-Robles-The 2012)

Let σ_a be a Schubert class in $G(k,n)$. Set $a_0 = 0$ and $a_{k+1} = \infty$. The Schubert class σ_a is multi-rigid if and only if for all essential sub-indices a_i,

$$a_{i-1} + 1 = a_i \leq a_{i+1} - 3.$$

Theorem

If $a_{i-1} + 1 = a_i \leq a_{i+1} - 3$, then for every irreducible representative X of $m\sigma_a$, $m \in \mathbb{Z}^+$, there exists a subspace F_{a_i} such that

$$\dim(F_{a_i} \cap \Lambda) \geq i, \forall \Lambda \in X$$

1. If $a_i \leq a_{i+1} - 3$, then there exists a variety Y of projective dimension $a_i - 1$ such that

$$\dim(Y \cap \mathbb{P}(\Lambda)) \geq i - 1, \forall \Lambda \in X$$

2. If furthermore $a_{i-1} + 1 = a_i$, then Y has to be linear.

- A sub-index satisfying the conditions in the theorem is called *multi-rigid*.

Let σ_{a^α} be a Schubert class for $F(d_1, \ldots, d_k; n)$.

- An essential sub-index a_i is called multi-rigid if for every irreducible representative X of $m\sigma_{a^\alpha}$, $m \in \mathbb{Z}^+$, there exists a subspace F_{a_i} of dimension a_i such that

$$\dim(F_{a_i} \cap \Lambda_j) \geq \mu_{i,j}, 1 \leq j \leq k, \forall \Lambda_\bullet \in X$$

Corollary

An essential sub-index is multi-rigid if it is multi-rigid with respect to $(\pi_t)_*(\sigma_{a^\alpha})$ for some $1 \leq t \leq k$.

- If $a_i = a_{i-1} + 1 = a_{i+1} - 2$ and $\alpha_i < \alpha_{i+1} \leq \alpha_{i-1}$, then a_i is multi-rigid.

$$\ldots, (a_i - 1)^2, a_i^1, (a_i + 2)^2, (a_i + 3)^1, \ldots$$

$$\begin{array}{ccc} \pi_1 \swarrow & & \searrow \pi_2 \\ \ldots, a_i, a_i + 3, \ldots & & \ldots, a_i - 1, a_i, a_i + 2, \ldots \\ \Downarrow & & \Downarrow \\ \exists Y & \rightsquigarrow & Y \text{ is linear} \end{array}$$

Theorem

An essential sub-index a_i is multi-rigid if and only if one of the following holds:

1. *a_i is multi-rigid with respect to $(\pi_t)_*(\sigma_{a^\alpha})$ for some $1 \leq t \leq k$;*
2. *$a_i = a_{i-1} + 1 = a_{i+1} - 2$ and $\alpha_i < \alpha_{i+1} \leq \alpha_{i-1}$.*

Corollary

A Schubert class σ_{a^α} for $F(d_1, ..., d_k; n)$ is multi-rigid if and only if all essential sub-indices are multi-rigid and the set of all essential sub-indices is strict totally ordered under the relation '$\rightarrow$'.

Orthogonal Grassmannians

Assume $n = \dim(V)$ is odd. Let q be a non-degenerate symmetric bilinear form on V. A linear subspace W is called *isotropic* with respect to q if $q(W, W) = 0$.
The orthogonal Grassmannian $OG(k, n)$ is the subvariety of $G(k, n)$ that parametrizes all isotropic subspaces of dimension k.

$$F_{a_1} \subset \dots \subset F_{a_s} \subset F^{\perp}_{b_{k-s}} \subset \dots \subset F^{\perp}_{b_1}$$

satisfying

- $a_s \leq [\frac{n}{2}]$;
- $b_{k-s} \leq [\frac{n}{2}] - 1$;
- $a_i - b_j \neq 1$.

Remark: If $n = 2k$ or $2k + 1$, the sequence $b_\bullet$ is uniquely determined by $a_\bullet$ and the relation $a_i \neq b_j + 1$.

$$\Sigma_{a;b} := \{\Lambda \in OG(k, n) | \dim(F_{a_i} \cap \Lambda) \geq i, \dim(F^{\perp}_{b_j} \cap \Lambda) \geq k - j + 1\}$$

- $OF(d_1, ..., d_k; n) := \{\Lambda_\bullet = (\Lambda_1, ..., \Lambda_k) | \Lambda_t \in OG(d_t, n)\}$

$$F_{a_1}^{\alpha_1} \subset ... \subset F_{a_s}^{\alpha_s} \subset (F_{b_{d_k-s}}^{\perp})^{\beta_{d_k-s}} \subset ... \subset (F_{b_1}^{\perp})^{\beta_1}$$
$$\Downarrow$$
$$F_\bullet^{\leq t}, 1 \leq t \leq k$$
$$\Downarrow$$
$$\Sigma(F_\bullet^{\leq t}) \subset OG(d_t, n)$$

$$\Sigma_{a^\alpha; b^\beta}(F_\bullet) := \{\Lambda_\bullet \in OF | \dim(F_{a_i} \cap \Lambda_t) \geq \mu_{i,t}, \dim(F_{b_j}^{\perp} \cap \Lambda_t) \geq \nu_{j,t}\}$$
$$\mu_{i,t} := \#\{p | p \leq i, \alpha_p \leq t\}$$
$$\nu_{j,t} := \mu_{s,t} + \#\{q | q \geq j, \beta_q \leq t\}$$

Assume $[X] \sim_r m\sigma_{a^\alpha;b^\beta}$, $m \in \mathbb{Z}^+$.

a_i is essential in t-th component, $\dim(F_{a_i} \cap \Lambda_t) \geq \mu_{i,t}$ for some t	$\Rightarrow$	$\dim(F_{a_i} \cap \Lambda_r) \geq \mu_{i,r}$ for all $1 \leq r \leq k$
b_j is essential in t-th component, $\dim(F_{b_j}^{\perp} \cap \Lambda_t) \geq \nu_{j,t}$ for some t	$\Rightarrow$	$\dim(F_{b_j}^{\perp} \cap \Lambda_r) \geq \nu_{j,r}$ for all $1 \leq r \leq k$

Corollary

Let $\sigma_{a;b}$ be a Schubert class for $OG(k,n)$. Let b' be the maximal admissible sequence associated to a. If a_i is rigid (or multi-rigid resp.) with respect to the Schubert class $\sigma_{a;b'}$ for $OG(\frac{n-1}{2},n)$, then a_i is also rigid (or multi-rigid resp.) with respect to the Schubert class $\sigma_{a;b}$.

Assume X is irreducible and $[X] \sim m\sigma_{a;b}$, $m \in \mathbb{Z}^+$. Consider the incidence correspondence

$$I := \{(\Lambda,\Lambda')|\Lambda \in X, \Lambda' \in OG(\frac{n-1}{2},n)\} \subset OF(k,\frac{n-1}{2};n).$$

The class of $\pi_2(I)$ is given by $m'\sigma_{a;b'}$ for some $m' \in \mathbb{Z}^+$. By assumption, there exists F_{a_i} such that $\dim(F_{a_i} \cap \Lambda') \geq i$ for all $\Lambda' \in \pi_2(I)$. Apply the previous result, $\dim(F_{a_i} \cap \Lambda) \geq i$ for all $\Lambda \in X$.

Thank you for your attention!