Erdős inequality for primitive sets

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1. Definition of primitive set

$A \subset \mathbb{N}$ is called primitive if for any $a_1, a_2 \in A$ such that $a_1 | a_2$, we have $a_1 = a_2$.
Example: $A = \{a, a+1, \dots 2a-1\}$.
Denote $\Omega(n)$ the number of prime divisors of n counted with multiplicity. That is $\Omega(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}) = \alpha_1 + \alpha_2 + \dots + \alpha_k$.
$\mathbb{P}_k := \{n \in \mathbb{N} : \Omega(n) = k\}$ is an example of primitive set.
$\mathbb{P} := \mathbb{P}_1$ is the set of primes.

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$\mathbb{P} := \mathbb{P}_1$ is the set of primes.
Denote $P(n)$ the largest prime divisor of n.

Theorem (Erdős, 1935)

For any primitive set A,

$$\sum_{a \in A} \frac{1}{a \log a} \leq M,$$

where M is an absolute constant. Also

$$\sum_{a \in A} \frac{1}{a} \prod_{p \leq P(a)} \left(1 - \frac{1}{p}\right) \leq 1.$$

2. Generalization of Erdős inequality

$$\prod_{p \le n} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log n}.$$

Hence

$$\sum_{a \in A} \frac{1}{a \log a} \le \sum_{a \in A} \frac{1}{a \log P(a)} \ll \sum_{a \in A} \frac{1}{a} \prod_{p \le P(a)} \left(1 - \frac{1}{p}\right) \le 1.$$

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Theorem (K, 2024)

Let $(\mathbb{P},\preceq)$ be the set of primes with some linear order. Let f be a completely multiplicative function such that $0\le f(p)\le 1$ for each prime. Denote $P'(n)$ the maximal prime divisor of n with respect to $\preceq$. Then for any primitive set A Then

$$\sum_{a\in A}f(a)\prod_{p\prec P'(a)}(1-f(p))\le 1. \qquad (1)$$

3. Erdős functions

Taking $f(p) = z/p$, where $0 < z < 2$, equation 1 implies that for each primitive set A

$$\sum_{a \in A} \frac{z^{\Omega(a)}}{a(\log a)^z} \le \sum_{a \in A} \frac{z^{\Omega(a)}}{a(\log P(a))^z} \ll \sum_{a \in A} \frac{z^{\Omega(a)}}{a} \prod_{p < P(a)} \left(1 - \frac{z}{p}\right) \le 1.$$

Denote

$$f_z(A) := \sum_{a \in A} \frac{z^{\Omega(a)}}{a(\log a)^z}$$

$$f_1(A) = \sum_{a \in A} \frac{1}{a \log a}.$$

$$U(z) := \sup_{A \text{ primitive}} f_z(A).$$

Now we know that $U(z)$ is bounded for each $0 < z < 2$.

4. $f_z(\mathbb{P}_k)$

How to compute $f_1(\mathbb{P}), f_1(\mathbb{P}_k)$ and $f_z(\mathbb{P}_k)$? The sum $\sum_p 1/(p \log p)$ converges very slowly. $\sum_{p<10^9} \frac{1}{p \log p} \approx 1.588$, while

$f_1(\mathbb{P}) = 1.636616323351260868569658003921863671181597076131 29\ldots$

(Henri Cohen, 1991)
Conjecture (Banks, Martin, 2013): $f_1(\mathbb{P}_k)$ is monotonically decreasing sequence.

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k	$f_1(\mathbb{P}_k)$
2	1.1448165734059179915
3	1.0308351017932175719
4	0.9973421485952523597
5	0.9888821921300755349
6	0.9887534530145096063
7	0.9910205950027380022
8	0.9935373386530404095
9	0.9956203792390954090
10	0.9971495172651382446

5. $f_z(\mathbb{P}_k)$

Theorem (Lichtman, 2019)

$\forall k \neq 6,\ f_1(\mathbb{P}_k) > f_1(\mathbb{P}_6)$.

$$\int_1^\infty n^{-s}(s-1)^{z-1}ds = \frac{\Gamma(z)}{n(\log n)^z} \quad \Rightarrow$$

$$f_z(\mathbb{P}_k) = \frac{z^k}{\Gamma(z)} \int_1^\infty P_k(s)(s-1)^{z-1}ds, \quad P_k(s) = \sum_{\Omega(n)=k} \frac{1}{n^s}.$$

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Theorem (Lichtman, 2019)

$$f_1(\mathbb{P}_k) = 1 + O(k^{-1/2+\delta}).$$

6. $f_z(\mathbb{P}_k)$

$$d_w := 2^{-w} \prod_{p>2} \left(1 - \frac{w}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{w}.$$

Theorem (Gorodetsky, Lichtman, Wong, 2023)

$$f_1(\mathbb{P}_k) = 1 - 2^{-k} \frac{d_2}{4} (\log 2)(k^2 + O(k \log k))$$

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If $0 < z < 2$, then

$$f_z(\mathbb{P}_k) \sim G(z), \qquad G(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{z}.$$

7. $f_z(\mathbb{P}_k)$

Theorem (K, 2024)

For $\varepsilon > 0$ *and* $z \in [\varepsilon, 1-\varepsilon]$

$$f_z(\mathbb{P}_k) = G(z) + \left(\frac{z}{z+1}\right)^k \frac{2d_{z+1}}{\Gamma(z)(1-z)} \left(\gamma - \frac{z\log 2}{1-z} - \sum_{p>2} \frac{z\log p}{(p-1)(p-z-1)}\right) + O_\varepsilon\left(\left(\frac{z}{2}\right)^k k^{2-z}\right).$$

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$$f_z(\mathbb{P}_k) = G(z) + O_\varepsilon\left(\left(\frac{z}{2}\right)^k k^{2-z}\right),$$

$$f_2(\mathbb{P}_k) = d_2(k - 2\log k + O(\sqrt{\log k})).$$

8. $f_z(\mathbb{P}_k)$

Proof:

$$f_z(\mathbb{P}_k) = \sum_{j=0}^{k} \gamma_{k,j}(z), \qquad \gamma_{k,j}(z) = \sum_{\substack{\Omega(n)=k \\ 2^j || n}} \frac{z^k}{n(\log n)^z},$$

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$$\gamma_{k,j}(z) = \frac{z^k}{\Gamma(z)} \int_1^\infty P_{k,j}(s)(s-1)^{z-1}\, ds, \quad P_{k,j}(s) := \sum_{\substack{\Omega(n)=k-j \\ (n,2)=1}} \frac{2^{-js}}{n^s},$$

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$$\gamma_{k,j}(z) \approx \frac{z^k}{\Gamma(z)} I_{k,j}, \quad I_{k,j} = \int_1^2 P_{k,j}(s)(s-1)^{z-1}\, ds.$$

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$$G_2(s,w) := (\zeta(s)(s-1))^w \left(1-\frac{1}{2^s}\right)^w \prod_{p>2}\left(1-\frac{w}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^w,$$

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$$I'_{k,j} = \int_1^2 2^{-js}[w^{k-j}]((s-1)^{z-w-1} G_2(1,w)) ds$$

9. $f_z(\mathbb{P}_k)$

$$I'_{k,j} = \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{G_2(1,w)}{w^{k-j+1}} \left(\int_1^2 2^{-js}(s-1)^{z-w-1} ds \right) dw.$$

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$$2^{-2j} \left(\frac{1}{z-w} + \frac{j \log 2}{(z-w)(z-w+1)} + \frac{(j \log 2)^2}{(z-w)(z-w+1)(z-w+2)} + \dots \right).$$

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Move the contour to $|w| = R = \min((k-j)/\log k, 2.5)$. The largest error comes from $R \approx 2$. This is worse, then $h_{j,z}(w)$ has a pole at 2. These is the case if $z = 1$ and $z = 2$ and then the error term is multiplied by $\sqrt{\log k}$.

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For $z < 1$ we take

$$G_2(s,w) \approx G_2(1,w) + (s-1)G'_2(1,w).$$

10. $U(z)$

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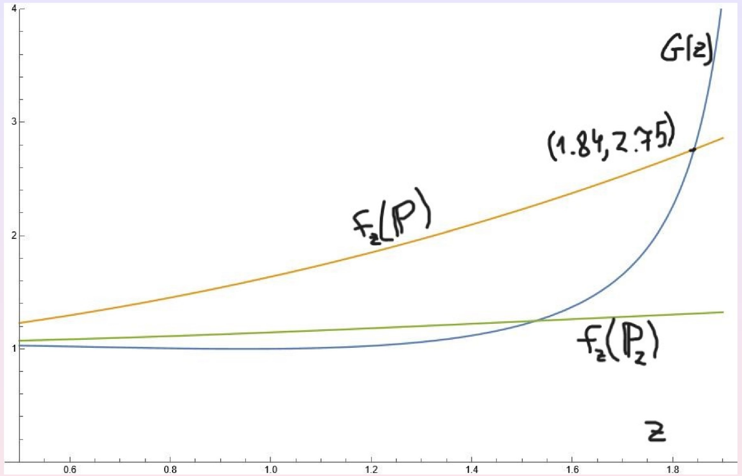
Erdős conjecture (1986): $U(1) = f_1(\mathbb{P})$.

Theorem (Lichtman, 2022)

$$U(1) = f_1(\mathbb{P}).$$

It is true that $U(z) = f_z(\mathbb{P})$ if z is in some neighborhood of 1.
Also obviously $U(z) \geq G(z) = \lim_{k\to\infty} f_z(\mathbb{P}_k)$ and $G(z)$ has a pole at 2.

11. $U(z)$



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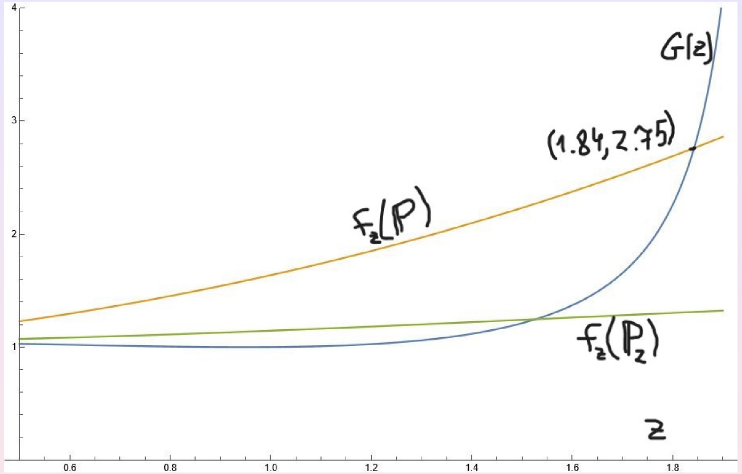

Conjecture 1

$U(z) = f_z(\mathbb{P})$ *for* $z \in (0, 1]$.

12. Primitive density

What can be said about $\lim_{z \to 0} f_z(A)$?

Definition

Let A be a set of natural numbers.

$$\overline{\eta}(A) := \limsup_{z \to 0} f_z(A), \quad \underline{\eta}(A) := \liminf_{z \to 0} f_z(A).$$

Let us call $\overline{\eta}(A)$ u $\underline{\eta}(A)$ upper and lower primitive densities of A. If $\overline{\eta}(A) = \underline{\eta}(A)$, we say that A has primitive density $\eta(A) := \overline{\eta}(A) = \underline{\eta}(A)$.

Note that $\eta(\mathbb{N}) = \infty$.

13. Primitive density

Theorem (K, 2024)

A) For any primitive A, $\overline{\eta}(A) \le 1$;
B) $\forall k \ge 1, \ \eta(\mathbb{P}_k) = 1$;
C) Let $A \subset \mathbb{P}_k$. We say that A has Dirichlet density c, if

$$\lim_{s \to 1+} \frac{\sum_{a \in A} a^{-s}}{\sum_{m \in \mathbb{P}_k} m^{-s}} = c$$

If A has Dirichlet density c, then it has primitive density c;

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$$\lim_{s \to 1+} \frac{\sum_{a \in A} a^{-s}}{\sum_{m \in \mathbb{P}_k} m^{-s}} = c$$

If A has Dirichlet density c, then it has primitive density c;
D) If $\overline{\eta}(B) > n-1,\ n \in \mathbb{N}$, then there are n elements $b_1, b_2, \ldots, b_n$, such that $b_i | b_{i+1}$ for all $1 \leq i < n$; This is not true for $n = \infty$.

14. Can we increase $f_z(A)$?

The sum of $1/(n \log n)$ over a primitive set is convergent. For any $\varepsilon > 0$ the sum of $1/(n(\log n)^{\varepsilon})$ over $\mathbb{P}_k$ converges for any $k \geq 1$ (but it is not uniformly bounded in k). We can ask whether we can replace $1/(n \log n)$ with some function which decays slower, but with the property, that the sum of this function over an arbitrary primitive set is convergent.

Theorem (K, 2024)

For each function $\psi(n)$ such that $\lim_{\Omega(n)\to\infty} \psi(n) = +\infty$ there exists a primitive set A, which satisfies two properties:
I. For any $0 < z < 2$

$$\sum_{a \in A} \frac{\psi(a) z^{\Omega(a)}}{a(\log a)^z} = \infty,$$

II. $f_z(A) = \infty$ for any $z \geq 2$.

Thank You!