On string functions of the generalized parafermionic theories, mock theta functions, and false theta functions

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(joint work with Eric Mortenson)

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  2. For $q = e^{2\pi i \tau}$, $|q| < 1$,
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  is a holomorphic *modular function* (modular form of weight 0) on the upper half-plane with respect to some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, for some modular anomaly $a \in \mathbb{C}$.

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- How to **classify** modular invariant representations of some infinite-dimensional Lie algebra?

Kac–Wakimoto conjecture

- Let us consider the case of the Kac–Moody algebra

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- For $\widehat{\mathfrak{g}} = A_1^{(1)}$ the Kac–Wakimoto conjecture is known to be *true*.
- What are *admissible highest weight representations of $A_1^{(1)}$* ?

Admissible highest weight representations for $A_1^{(1)}$

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- In this paper we will consider only the case of $k = 0$, so that the spin, the coefficient of Λ_1 of λ is equal to ℓ and hence is a positive integer.

The modularity of the character

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$$\chi_\ell^N(z;q) := \mathrm{Tr}_{L(\lambda)}\left(q^{s_\lambda - d} z^{-J_0^3/2}\right),$$

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- Kac and Wakimoto proved the *Weyl–Kac character formula* for admissible highest weights λ

$$\chi_\ell^N(z;q) = \frac{\sum_{\sigma=\pm 1} \sigma\Theta_{\sigma(\ell+1),p'}(z;q^p)}{\sum_{\sigma=\pm 1} \sigma\Theta_{\sigma,2}(z;q)},$$

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- Kac and Wakimoto showed that the characters form a vector-valued Jacobi form.

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- From the definition we have the *Fourier expansion*

$$\chi_\ell^N(z,q) = \sum_{m\in 2\mathbb{Z}+\ell} C_{m,\ell}^N(q) q^{\frac{m^2}{4N}} z^{-\frac{1}{2}m}.$$

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- The **parafermionic partition function** over the Hilbert subspace of states, which are obtained from the highest weight state of Z_{2N} charge ℓ and which have Z_{2N} charge charge m, is

$$\mathcal{Z}^{\mathrm{PF}}_{m,l}(q) = \eta(q) C^N_{m,\ell}(q),$$

where we denote *Dedekind eta-function* as

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- Note that in the case of a fractional-level, such generalized PF theories are non-unitary.

Computation of string functions

- Kac and Peterson gave several examples of elegant evaluations in terms of theta functions of string functions of integrable highest weight representations of $A_1^{(1)}$,

$$c_{01}^{01} = \eta(\tau)^{-1},$$
$$c_{11}^{11} = \eta(\tau)^{-2}\eta(2\tau),$$
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- Kac and Peterson appeal to modularity to prove the string function identities. Specifically, they use the transformation law for string functions under the full modular group, together with the calculation of the first few terms in the Fourier expansions of the string functions.

Mock theta functions

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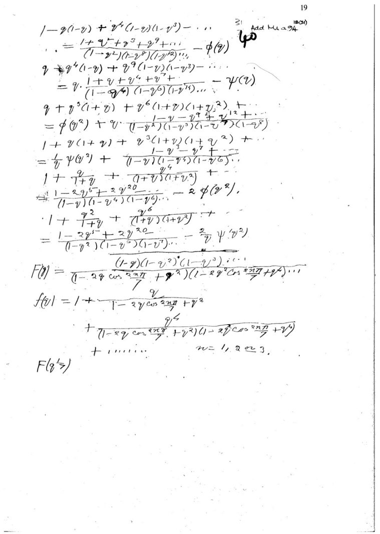
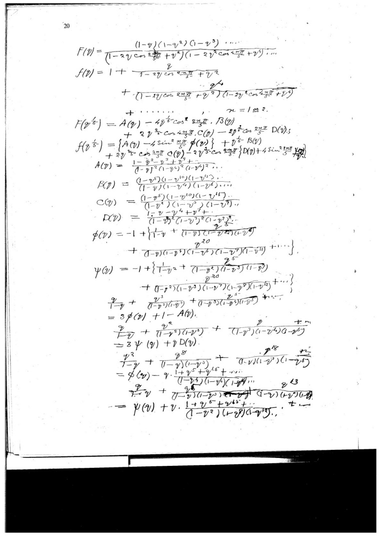
- As an example we present Ramanujan's classical second-order mock theta functions

$$\mu(q) := \sum_{n\geq 0}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2} \quad \text{and} \quad A(q) := \sum_{n\geq 0}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2},$$

both of which appearing in the "Lost Notebook" on page 8.

Lost Notebook





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  which were unified in Zwegers' celebrated thesis in 2002.
- We define the building blocks of Hecke-type double-sum form,

$$f_{a,b,c}(x,y;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

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- So for admissible level $N = p'/p - 2$, $0 \leq \ell \leq p' - 2$, $m \in 2\mathbb{Z} + \ell$ we can write

$$(q)_\infty^3 \mathcal{C}_{m,\ell}^N(q) = f_{1,p',2pp'}(q^{1+\frac{m+\ell}{2}}, -q^{p(p'+\ell+1)}; q) - f_{1,p',2pp'}(q^{\frac{m-\ell}{2}}, -q^{p(p'-(\ell+1))}; q).$$

The 1/2-level string functions

- We introduce the **theta function**

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- We can prove the following result for the string functions of the level $N = 1/2$.

Theorem (B., Mortenson, 2024)

Let $(p, p') = (2, 5)$, $0 \leq \ell \leq 3$ and $m \in 2\mathbb{Z} + \ell$. We have that

$$\mathcal{C}^{1/2}_{m,\ell}(q) - \frac{q^{\frac{1}{2}(m-\ell)} j(q^{1+\ell}; q^5)}{(q;q)^3_\infty} \left(\frac{1}{2}(-1)^m q^{\binom{m}{2}} \mu(q) + \sum_{k=0}^{m-1} (-1)^k q^{mk - \binom{k+1}{2}} \right)$$

is weight $-1/2$ weakly holomorphic modular form on subgroup $\Gamma_1(200) \subset \mathrm{SL}_2(\mathbb{Z})$.

Mock theta conjectures

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$$f_0(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n} = 2 - 2\sum_{n\geq 0} \frac{q^{10n^2}}{(q^2;q^{10})_{n+1}(q^8;q^{10})_n} + \frac{J_5 J_{5,10}}{J_{1,5}},$$

where we used the notation for specializations of theta function

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- Many identities in the Lost Notebook express mock theta functions in terms of single quotients of theta functions, with there being no apparent explanation for the phenomenon.

Mock theta conjectures-like identities for string functions

Theorem (B., Mortenson, 2024)

In terms of Ramanujan's second-order mock theta function $\mu(q)$, we have

$$(q)_\infty^3 \mathcal{C}_{0,0}^{1/2}(q) = \frac{1}{2} j(q;q^5)\mu(q) + \frac{1}{2}\cdot\frac{J_1^3 J_{10}^3}{J_4 J_5}\cdot\frac{1}{J_{1,10}J_{8,20}},$$

and

$$(q)_\infty^3 \mathcal{C}_{0,2}^{1/2}(q) = \frac{1}{2q} j(q^2;q^5)\mu(q) - \frac{1}{2q}\cdot\frac{J_1^3 J_{10}^3}{J_4 J_5}\cdot\frac{1}{J_{3,10}J_{4,20}}.$$

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Corollary (B., Mortenson, 2024)

In terms of Ramanujan's second-order mock theta function $A(q)$, we have

$$(q)_\infty^3 \mathcal{C}_{0,0}^{1/2}(q) = -2 j(q;q^5) A(-q) + \frac{J_1^4 J_4 J_{8,20}}{J_2^4},$$

and

$$(q)_\infty^3 \mathcal{C}_{0,2}^{1/2}(q) = -\frac{2}{q} j(q^2;q^5) A(-q) - \frac{J_1^4 J_4 J_{4,20}}{J_2^4}.$$

The mixed mock modular transformations of 1/2-level string functions

Theorem (B., Mortenson, 2024)

We have

$$\begin{pmatrix} C_{0,0}^{1/2} \\ C_{0,2}^{1/2} \end{pmatrix}(\tau+1) = \begin{pmatrix} \zeta_{40}^{-1} & 0 \\ 0 & \zeta_{40}^{-9} \end{pmatrix} \begin{pmatrix} C_{0,0}^{1/2} \\ C_{0,2}^{1/2} \end{pmatrix}(\tau)$$

and

$$\begin{pmatrix} C_{0,0}^{1/2} \\ C_{0,2}^{1/2} \end{pmatrix}(\tau) = \sqrt{-i\tau} \cdot \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\left(\frac{2\pi}{5}\right) & -\sin\left(\frac{\pi}{5}\right) \\ -\sin\left(\frac{\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) \end{pmatrix} \begin{pmatrix} C_{0,0}^{1/2} \\ C_{0,2}^{1/2} \end{pmatrix}\left(-\frac{1}{\tau}\right)$$

$$- \frac{i}{2} \cdot \frac{1}{\eta(\tau)^3} \cdot \begin{pmatrix} \mathcal{J}_{1,5} \\ \mathcal{J}_{2,5} \end{pmatrix}(\tau) \cdot \int_0^{i\infty} \frac{\eta(w)^3}{\sqrt{-i(w+\tau)}} dw.$$

False theta functions and partial theta functions

- **False theta functions** are theta functions but with the "wrong signs" firstly considered by Rogers. Let $r \in \mathbb{Z}$ and define

$$\mathrm{sg}(r) := \begin{cases} 1, & \text{if } r \geq 0 \\ -1, & \text{if } r < 0. \end{cases}$$

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- Then, if we write the theta function of definition with incorrect signs we get

$$\sum_{n\in\mathbb{Z}} \mathrm{sg}(n)(-1)^n q^{\binom{n+1}{2}} = \sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}} - \sum_{n=-\infty}^{-1}(-1)^n q^{\binom{n+1}{2}} = 2\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}},$$

that is, it turns out to be a **partial theta function**, i.e. functions resembling theta functions but with the summation over $\mathbb{Z}$ replaced by a partial lattice (e.g. $n \geq n_0$ for $n_0 \in \mathbb{Z}$).

The $(-1/2)$-level string functions

We give a new proof of the result on the $(-1/2)$-level string functions.

Theorem (Schilling, Warnaar, 2002)

Let $(p, p') = (2, 3)$, $\ell \in \{0, 1\}$ and $m \in 2\mathbb{Z} + \ell$. We have

$$\mathcal{C}^{-1/2}_{m,\ell}(q) = \frac{q^{\frac{1}{2}(m-\ell)}}{(q)^2_\infty} \sum_{i \geq 0} (-1)^i q^{\frac{1}{2}i(i+2m+1)}.$$

The $(-2/3)$-level string functions

We also obtain an evaluation for the $(-2/3)$-level string functions.

Theorem (B., Mortenson, 2024)

Let $(p,p')=(3,4)$, $0\le \ell\le 2$ *and* $m\in 2\mathbb{Z}+\ell$. *We have*

$$\begin{aligned}
\mathcal{C}_{m,\ell}^{-2/3}(q)=\frac{1}{2(q)_\infty^3 J_{16}}\Big(& j(q^{1+\ell};q^8)j(q^{10+2\ell};q^{16})\cdot q^{\frac{1}{2}(m-\ell)}\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)q^{r(6r+3m+2)}\\
&+j(q^{5+\ell};q^8)j(q^{2+2\ell};q^{16})\cdot q^{(2m-\ell)+3}\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)q^{r(6r+3m+8)}\\
&-j(q^{5+\ell};q^8)j(q^{2+2\ell};q^{16})\cdot q^{(m-\ell)+1}\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)q^{r(6r+3m+4)}\\
&-j(q^{1+\ell};q^8)j(q^{10+2\ell};q^{16})\cdot q^{\frac{1}{2}(5m-\ell)+4}\sum_{r\in\mathbb{Z}}\operatorname{sg}(r)q^{r(6r+3m+10)}\Big).
\end{aligned}$$

Thank you for your attention!

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