Explicit / effective birational geometry

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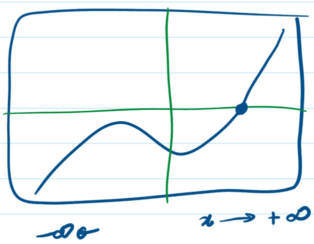
Effective / explicit / constructive maths:

- algorithmic
- classification
- computational maths
- effective results are more difficult to prove

Exa: $f \in \mathbb{R}[t]$ of deg 3, monic

$\exists$ root of f

as $f(x) \to \pm\infty$ when $x \to \pm\infty$



- effective version: $f = x^3 + ax^2 + bx + c$

can write down the roots in terms of a, b, c

Exa: Hilbert Nullstellensatz

- $f_1, \dots, f_r \in \mathbb{C}[t_1, \dots, t_n]$

$V(f_1, \dots, f_r) =$ set of common solutions of $f_i = \phi \implies \exists\, g_i \in \mathbb{C}[t_1, \dots, t_n]$

• $f_1, \dots, f_r \in \mathbb{C}[t_1, \dots, t_n]$

$V(f_1, \dots, f_r)$ = set of common solutions of $f_i = \emptyset$ $\Longrightarrow$ $\exists\, g_i \in \mathbb{C}[t_1, \dots, t_n]$

s.t. $f_1 g_1 + \dots + f_r g_r = 1$.

• effective; if $\deg f_i \leq \underline{d}$ $\Rightarrow$ $\exists\, g_i$ of $\deg \leq \max\{3, d\}^{\min\{r, n\}}$

Exa: resolution of singularities (char $= 0$)

• X variety, $\exists$ resolution $g: W \to X$ $\begin{cases} g \text{ birational, proj morphism} \\ W \text{ smooth} \end{cases}$

• effective: X proj, $X \subseteq \mathbb{P}^n$ given by $F_1, \dots, F_r$ of $\deg \leq d$

$\exists$ resolution $g: W \to X$ s.t.

$W \subseteq \mathbb{P}^m$ given by $G_1, \dots, G_s$ of $\deg \leq e$. [Bierstone-Milman, et al]

m, e can be computed in terms of n, d.

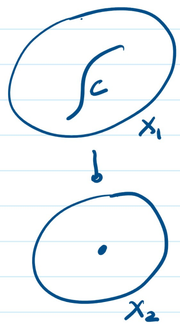
Exa: minimal model program

• surfaces: X smooth proj surface

$X = X_1 \to X_2 \to \cdots \to X_n = Y$ (MMP)

If $\exists$ $C \subseteq X_1$ with $C \simeq \mathbb{P}^1$ & $C^2 = -1$, then $X_1 \to X_2$ contraction of C

If --- $\subseteq X_2$ - - - - - .



$\begin{cases} K_Y \text{ is nef: } K_Y \cdot D \geq 0 \ \forall \text{ curve } D \subseteq Y, \text{ or} \\ \exists \text{ Fano fib } Y \to T \ \begin{cases} Y = \mathbb{P}^2 \to T = pt \\ Y \to T \ \ \mathbb{P}^1\text{-bundle over curve } T \end{cases} \end{cases}$

• effective: • $\rho(X) \leq d$ $\Rightarrow$ # steps in MMP is at most d.

• $X \subseteq \mathbb{P}^n$ given by $F_1, \dots, F_r$ of $\deg \leq d$ $\Rightarrow$ $Y \subseteq \mathbb{P}^m$ given by $G_1, \dots, G_s$ of $\deg \leq e$.

(n, d fixed $\Rightarrow$ m, e can be computed in terms of n, d).

• higher dimension ? e.g. $\dim = 3$.

Exa: Fujita conjecture: X smooth proj var of dim d, A ample divisor on X.

- $K_X + mA$ globally generated, very ample. $m >> 0$
- Conj: $K_X + mA$ is globally gen when $m \geq d+1$
  very ample — $m \geq d+2$.

Exa: General type varieties

X smooth proj var of general type: Kodaira dim $\kappa(X) = \dim X = d$.

- $|mK_X|$ def bir map, $\forall m >> 0$.
- $d=1$: K_X may or may not be very ample
  $3K_X$ is very ample, $|3K_X|$ defines a bir map $X \dashrightarrow \mathbb{P}^n$
- $d=2$: $|5K_X|$ def a bir map: $H^0(X, 5K_X)$, choose basis $h_0, \ldots, h_n$
  $X \dashrightarrow \mathbb{P}^n$ given by $h_0, \ldots, h_n$.
- $d=3$: $|73K_X|$ def a bir map. [J.A. Chen, M. Chen]
- for $d \geq 4$; $\exists\, m$ depend. on d s.t. $|mK_X|$ def a bir map. [Hacon-McKernan, Takayama, Tsuji]
  Conj: we can find effective m (in terms of d).
- Conj: $\exists$ effective $\alpha > 0$ (in terms of d) s.t. $\mathrm{vol}(K_X) \geq \alpha$.

  (for $d=1$, $\mathrm{vol} = \deg$)
  (when K_X is ample, $\mathrm{vol}(K_X) = K_X^d$)

Exa: Fano varieties

- X smooth Fano var of dim d, $\exists$ effective m s.t. $|-mK_X|$ very ample.
  [Fano, Kollár-Miyaoka-Mori]
- X Fano var of dim $=d$ with ε-lc singularities, $\varepsilon > 0$.
  $\exists\, m$ depend. on d, ε s.t. $|-mK_X|$ def a bir map $X \dashrightarrow Y \subseteq \mathbb{P}^n$ (even embedding) [Birkar]
- Conj: $\exists$ effective m, n (in terms of d, ε). (known $d=2$).

- X Fano var, klt singularities.

  $\exists$ m depend. on d s.t. $H^0(X, -mK_X) \neq 0$. [Birkar]

- Conj: $\exists$ effective m (in terms of d). (d=3 partially known) [Birkar - J. Liu]

Exa: Calabi-Yau varieties

- X smooth Calabi-Yau var, A ample div. ($K_X \equiv 0$)

  $\exists$ effective m s.t. $|mA|$ def bir map.

- X Calabi-Yau with klt singularities, A ample div. [Birkar]

  $\exists$ m deped. on d s.t. $|mA|$ def bir map.

- Conj: $\exists$ effective m (in terms of d). ($d=3$ partially known) [C. Jiang]

- Conj (Yau):

  $\{$ smooth proj Calabi-Yau var X of $\dim = d\}$

  is topologically bounded.

- Effective version? Effective bound on Betti numbers?

Exa: Fano fibrations

$X \to Z$ Fano fib ($-K_X$ ample $/Z$), X ε-lc singularities, $\varepsilon > 0$, $\dim X/Z = d$.

$\exists$ $\delta > 0$ depend. on d, ε s.t. Z is δ-lc. [Birkar]

- Conj: $\exists$ effective δ (in terms of d, ε).

  [$d=1$, B. Chen]

  [$d=2$, partially known, Mori-Prokhorov]

Exa: Moduli divisors

$X \xrightarrow{f} Z$, X klt singularities, $K_X \sim_{\mathbb{Q}} f^*L$. (CY fib)

Canonical bundle formula:

$$K_X \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

$$K_X \sim_{\mathbb{Q}} f^*(\underbrace{K_Z + B_Z}_{\text{disc. div}} + \underbrace{M_Z}_{\text{moduli div}})$$

- Conj (Prokhorov–Shokurov)
  - $\exists$ bir model $Z' \to Z$ s.t. $M_{Z'}$ is semi-ample,
  - moreover, $\exists$ effective m (in terms of $\dim X/Z$) s.t. $|mM_{Z'}|$ is base point free.

Exa: Group actions

G finite group, abelian index $A(G) = \min\{[G:L] \mid L \trianglelefteq G \text{ normal abelian}\}$

- G acts on rationally-connected proj var of dim d
  $\Rightarrow \exists\, J$ deped. on d s.t. $A(G) \leq J$. [Prokhorov–Shramov, Birkar]
- Conj: $\exists$ effective J (in terms of d). [$d=3$, Prokhorov–Shramov]
- G acts on a non-uniruled proj var X
  $\Rightarrow \exists\, J$ deped. on X s.t. $A(G) \leq J$. [Prokhorov–Shramov]
- Conj: $\exists$ effective J (in terms of X)

Rem: With advances in recent years we can approach some of these problems.

Thm (Birkar–Liu). Assume X is a non-exceptional Fano 3-fold with klt singularities. Then $\exists$ m-complement for some

$$m \leq 192 \times (42)! \times 84^{128 \times 42^5}.$$

In particular,

$$H^0(X, -mK_X) \neq 0.$$

Proof sketch:

Since X is exceptional, using MMP, can replace X and assume

Proof sketch:

Since X is exceptional, using MMP, can replace X and assume

(X, S) is dlt & $-(K_X+S)$ is semi-ample. (for simplicity, say plt)

case: $\kappa(-(K_X+S)) = 3$.

Using vanishing of cohomology, can lift a complement from S and reduce to $\dim = 2$.

case: $\kappa(-(K_X+S)) = 2$:

Here $-(K_X+S)$ induces a fibration $X \to Z$, $\dim Z = 2$.

For any horizontal comp $T \subseteq S$, stein deg $(T/Z) \leq 2$.

Construct a complement on T, descend to Z and pullback to X.

case: $\kappa(-(K_X+S)) = 1$:

Here $-(K_X+S)$ induces a fibration $X \xrightarrow{f} Z$, $\dim Z = 1$.

Write the canonical bundle formula

$$K_X + S \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z).$$

But we need to control the coefficients of B_Z, M_Z.

And then construct a complement for $K_Z + B_Z + M_Z$ and pull back to X.

case: $\kappa(-(K_X+S)) = 0$:

Using MMP and above arguments, one reduces to the case $\rho(X) = 1$.

Here using vanishing thms, we can lift a complement from S.