Categorical Resolutions via Toric Exoflops

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5 November 2024

Overview

Motivation

- Mirror Symmetry: Two Calabi-Yau varieties X, Y are mirror if the complex algebraic structure on X is replicated by the symplectic structure on Y and vice-versa.
- Different ways to express structure may yield different constructions/
- Homological Mirror Symmetry:

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \simeq \operatorname{Fuk}(Y)$$

- For distinct mirrors X, X' of Y would thus expect $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X')$.

Libgober-Teitelbaum mirror

Libgober and Teitelbaum proposed the following mirror construction. Let

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5, \quad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2,$$

and consider the complete intersection $V_\lambda = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5$.

The proposed mirror $W_{LT,\lambda}$ to V_λ is a (minimal) Calabi-Yau resolution of the variety $V_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5 / G_{81}$, where $G_{81} \leq PGL(5, \mathbb{C})$ is a specified order 81 subgroup. They provide topological evidence of mirror symmetry by showing

$$\chi(V_\lambda) = -\chi(W_{LT,\lambda}).$$

This is different from the mirror one would obtain by the Batyrev-Borisov construction. However, the two mirrors are indeed derived equivalent (M., '23). The proof uses the machinery of toric exoflops.

An exoflop

$$
\begin{array}{ccc}
\mathcal{Z} & \xleftrightarrow{LG/CI} & (X, G, W) \\
 & & \big\uparrow\big\downarrow \, exo \\
 & & (\bar{X}, \bar{G}, \bar{W}) \\
 & & \big\updownarrow \, flop \\
\mathcal{Z}' & \xleftrightarrow{LG/CI} & (X', G', W')
\end{array}
$$

Exo: Partially compactify the relevant space X in a gauged LG model while extending the group action G and the global function W.

Flop: A birational transformation of the partial compactification of X, determined by varying the stability parameter of a prescribed geometric invariant theory.

A naive hope

We hoped this would straightforwardly generalize the Libgober-Teitelbaum construction. Unfortunately, if the complete intersection is not smooth, the exoflop no longer yields equivalences.

$$Q_{1,n,\lambda} = x_1^n + x_2^n + \cdots + x_n^n - \lambda x_{n+1}x_{n+2}\ldots x_{2n},$$

$$Q_{2,n,\lambda} = x_{n+1}^n + x_{n+2}^n + \cdots + x_{2n}^n - \lambda x_1 x_2 \ldots x_n.$$

$$G_n \cong (\mathbb{Z}/n\mathbb{Z})^{2(n-2)} \times (\mathbb{Z}/n^2\mathbb{Z}).$$

Categorical resolutions

Definition

Let $\tilde{\mathcal{D}}$ be the homotopy category of a homologically smooth and proper pretriangulated dg category. A pair of exact functors

$$F : \tilde{\mathcal{D}} \to \mathcal{D}$$
$$G : \mathcal{D}^{\mathrm{perf}} \to \tilde{\mathcal{D}}$$

is a categorical resolution of singularities if G is left adjoint to F and the natural morphism of functors $\mathrm{Id}_{\mathcal{D}^{\mathrm{perf}}} \to FG$ is an isomorphism. We say the categorical resolution is crepant if G is right adjoint to F.

Absolute derived categories

Let X be a smooth variety over $\mathbb{C}$, G an affine algebraic group acting on X and $W \in \Gamma(X, \mathcal{L})^G$ for a G-equivariant invertible sheaf $\mathcal{L}$. (X, G, W) is the data of a Landau-Ginzburg model. We can associate to it a absolute derived category $\mathrm{D}^{\mathrm{abs}}[(X, G, W)]$ (think equivalent of $\mathrm{D}^b_G(\operatorname{coh} X)$).

Exo

Let $\Sigma \subseteq N_{\mathbb{R}}$ a complete fan. $D_i = \sum_{\rho \in \Sigma(1)} a_{\rho i} D_\rho$ a torus invariant Weil divisor. $H_i = \{(m, \delta_{1i}, \ldots \delta_{ri})\}$.

$$P_{D_i} = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_{\rho i} \text{ for all } \rho \in \Sigma(1)\}.$$

Global sections of each divisor are

$$f_i = \sum_{m \in P_{D_i} \cap M} c_m \prod_{\rho \in \Sigma(1)} x_\rho^{\langle m, u_\rho \rangle + a_{\rho i}} \in \Gamma(X_\Sigma, \mathcal{O}(D_i))$$

and correspond to the global functions on tot $\bigoplus_{i=1}^{r} \mathcal{O}(-D_i)$

$$s_i = u_i \sum_{(m, b_1, \ldots, b_r) \in H_i \cap (M \times \mathbb{Z}^r)} c_m \prod_{\rho \in \Sigma(1)} x_{\bar{\rho}}^{\langle m, u_\rho \rangle + a_{\rho i}}$$

Abusing notation, write $s_i = u_i f_i$.

$$[\mathcal{Z}/G] = [Z(f_1, \ldots, f_r)/S_\Sigma] \subseteq [U_\Sigma/S_\Sigma]$$

and $W = \sum_{i=1}^r u_i f_i$.

Recall the R-charge action of $\mathbb{G}_m$ acting on a vector bundle by fiberwise dilation and consider projection character $\chi : S_\Sigma \times \mathbb{G}_m \to \mathbb{G}_m$. Note

$$W \in \Gamma(U_{\Sigma_{-D_1,\dots,-D_r}}, \mathcal{O}_{U_{\Sigma_{-D_1,\dots,-D_r}}}(\chi))^{S_{\Sigma_{-D_1,\dots,-D_r}} \times \mathbb{G}_\mathrm{m}}.$$

There exists an equivalence of categories

$$\Omega : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}/G]) \xrightarrow{\sim} \mathrm{D}^{\mathrm{abs}}[U_{\Sigma_{-D_1,\dots,-D_r}}, S_{\Sigma_{-D_1,\dots,-D_r}}(1) \times \mathbb{G}_{\mathrm{m}}, W],$$

where the $\mathbb{G}_{\mathrm{m}}$ acts with weights 0 on the coordinates x_ρ and 1 on the u_i. This result in its most general form is due to Hirano. So the complete intersection has LG model:

$$(U_{\Sigma_{-D_1,\dots,-D_r}}, S_{\Sigma_{-D_1,\dots,-D_r}}(1) \times \mathbb{G}_{\mathrm{m}}, W).$$

Assume from now Σ simplicial. Given the global function $W : \mathrm{tot} \bigoplus_{i=1}^{r} \mathcal{O}(-D_i) \to \mathbb{A}^1$, we can define

$$\Xi_{i,W} := \{\bar{m} = (m, \delta_{1i}, \ldots, \delta_{ri}) \in H_i \mid c_m \neq 0\}.$$

Let $\Xi_W = \bigcup X_{i,W}$ and $\sigma_W = \mathrm{Cone}(\Xi_W) \subseteq |\Sigma_{-D_1,\ldots,-D_r}|^\vee$. Take σ' so that $|\Sigma_{-D_1,\ldots,-D_r}| \subseteq \sigma' \subseteq \sigma_W^\vee$.

Lemma

There exists a simplicial fan Ψ *with support* σ' *so that* $\Sigma_{-D_1,\dots,-D_r}$ *is a subfan of* Ψ.

By Favero-Kelly, we have a stack isomorphism φ

$$[U_{\Sigma_{-D_1,\dots,-D_r}} \times {\mathbb{G}_m}^{\Psi(1)\setminus\Sigma_{-D_1,\dots,-D_r}(1)} / S_\Psi] \xrightarrow{\sim} [U_{\Sigma_{-D_1,\dots,-D_r}} / S_{\Sigma_{-D_1,\dots,-D_r}}],$$

which induces an equivalence of categories on the associated absolute derived categories.

Consider the S_Ψ-equivariant open immersion

$$i : U_{\Sigma_{-D_1,\dots,-D_r}} \times \mathbb{G}_m{}^{\Psi(1)\setminus\Sigma_{-D_1,\dots,-D_r}(1)} \hookrightarrow U_\Psi.$$

Note we can extend W to $\bar{W} \in \Gamma(U_\Psi, \mathcal{O}_{U_\Psi}(\chi))^{S_\Psi \times \mathbb{G}_m}$.

Theorem

Suppose $\mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}]$ *is homologically smooth and proper. Then we have the following crepant categorical resolution*

$$\begin{aligned} i_* \circ \varphi^* \circ \Omega : \mathrm{Perf}[\mathcal{Z}/G] &\longrightarrow \mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}]; \\ \Omega^{-1} \circ \varphi_* \circ i^* : \mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}] &\longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}/G]); \end{aligned} \tag{1}$$

Corollary

Consider the situation of Theorem above. Assume further that $[\mathcal{Z}/G]$ is a smooth Deligne-Mumford stack. Then $i_* \circ \varphi^* \circ \Omega$ is a fully faithful functor.

Corollary

Suppose we are in the situation of Corollary above. Assume further that $\mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}]$ is Calabi-Yau. Then $i_* \circ \varphi^* \circ \Omega$ is an equivalence.

Flop

Let $\sigma \subseteq N_{\mathbb{R}}$ be a $\mathbb{Q}$-Gorenstein cone and $\nu \subseteq \sigma \cap N$ be a finite, geometric collection of lattice points which contains the (primitive) ray generators of σ. Partition the set ν into two subsets

$$\begin{aligned} \nu_{=1} &= \{v \in \nu \mid \langle m_\sigma, v \rangle = 1\} \text{ and} \\ \nu_{\neq 1} &= \{v \in \nu \mid \langle m_\sigma, v \rangle \neq 1\}. \end{aligned} \tag{2}$$

Theorem (Theorem 5.8 of [FK18])

Let $\Psi, \tilde{\Sigma}$ *be simplicial fans such that* $\Psi(1) = \{ Cone(v) \mid v \in \nu \}$, $X_\Psi, X_{\tilde{\Sigma}}$ *semiprojective,* $\tilde{\Sigma}(1) \subseteq \nu_{=1}$, *and* $\mathrm{Cone}(\tilde{\Sigma}(1)) = |\Psi|$.

1. *If* $\langle m_\sigma, a \rangle > 1$ *for all* $a \in \nu_{\neq 1}$, *then there is a fully-faithful functor*

$$\mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}} \times \mathbb{G}_{\mathrm{m}}{}^{\nu \setminus \tilde{\Sigma}(1)}, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}] \to \mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}].$$

2. *If* $\langle m_\sigma, a \rangle < 1$ *for all* $a \in \nu_{\neq 1}$, *then there is a fully-faithful functor*

$$\mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}] \to \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}} \times \mathbb{G}_{\mathrm{m}}{}^{\nu \setminus \tilde{\Sigma}(1)}, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}].$$

3. *If* $\nu_{\neq 1} = \varnothing$, *then there is an equivalence*

$$\mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}} \times \mathbb{G}_{\mathrm{m}}{}^{\nu \setminus \tilde{\Sigma}(1)}, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}] \cong \mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}].$$

Take a fan $\tilde{\Sigma}$ as above. Suppose there exist rays $\rho'_1, \dots, \rho'_r \in \tilde{\Sigma}(1)$ with primitive generators $e_{\rho'_1}, \dots, e_{\rho'_r} \in N \times \mathbb{Z}^r$ so that

1. the induced projection

$$\pi : N \times \mathbb{Z}^r \to (N \times \mathbb{Z}^r)/(\oplus_{i=1}^t \mathbb{Z} \cdot e_{\rho'_i})$$

   induces the toric morphism $\pi : X_{\Sigma'_{-D'_1,\dots,-D'_r}} \to X_{\Sigma'}$ and this toric morphism is a rank r vector bundle whose sheaf of sections is $\oplus_{i=1}^r \mathcal{O}_{X_{\Sigma'}}(-D'_i)$, and
2. $e_{\rho'_1} + \cdots + e_{\rho'_r} = e_1 + \cdots + e_r \in \mathbb{N} \times \mathbb{Z}^r$.

Write the function $\bar{W}$ as

$$\bar{W} = \sum_{\bar{m} \in \Xi_{i,W}} c_m \prod_{\rho \in \tilde{\Sigma}(1)} x_\rho^{\langle \bar{m}, u_\rho \rangle}.$$

Then, there exists a partition $\Xi_{i,W} = H'_1 \cup \cdots \cup H'_r$ so that we can write

$$\bar{W} = u'_1 g_1 + \cdots + u'_r g_r, \text{ where } g_i = \sum_{\bar{m} \in H'_i} \prod_{\rho \in \tilde{\Sigma}(1) \setminus \{\rho'_1, \dots, \rho'_r\}} x_\rho^{\langle \bar{m}, u_\rho \rangle}$$

where $g_i \in \Gamma(X_{\Sigma'}, \mathcal{O}_{X_{\Sigma'}}(D'_i))$. Then we have the quotient stack

$$[\mathcal{Z}'/G'] := [Z(g_1, \dots, g_r)/S_{\tilde{\Sigma}(1)}] \subseteq [U_{\tilde{\Sigma}}/S_\Sigma] \qquad (3)$$

where

$$\mathrm{D^{abs}}[U_{\tilde{\Sigma}} \times {\mathbb{G}_\mathrm{m}}^{\nu \setminus \tilde{\Sigma}(1)}, G_\Psi \times \mathbb{G}_\mathrm{m}, \bar{W}] \cong \mathrm{D^b}(\mathrm{coh}[\mathcal{Z}'/G']).$$

If $[\mathcal{Z}'/G']$ is smooth, then $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}'/G'])$ is homologically smooth and proper. Also, if it is a CY orbifold, the category is CY. Combining all of these bits together gives us the exoflop!

Some results

Assume conditions from a few slides ago hold.

(a) If we are in either cases (a) or (c) of Theorem, and $[\mathcal{Z}'/G']$ is smooth, then we have that $\mathrm{D}^{\mathrm{abs}}[U_\Psi, G_\Psi \times \mathbb{G}_{\mathrm{m}}, \bar{W}]$ is homologically smooth and proper, hence $i_* \circ \varphi^* \circ \Omega$ and $\Omega^{-1} \circ \varphi_* \circ i^*$ form a categorical resolution for $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}/G])$.

(b) If we are in either cases (b) or (c) of Theorem, and $[\mathcal{Z}/G]$ and $[\mathcal{Z}'/G']$ are smooth, then we have a fully faithful functor

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}/G]) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}'/G']).$$

(c) If we are in case (c) of Theorem , $[\mathcal{Z}/G]$ is smooth, and $[\mathcal{Z}'/G']$ is a smooth Calabi-Yau orbifold, then we have an equivalence

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}/G]) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[\mathcal{Z}'/G']).$$

A second kind of resolution

Apart from Kuznetsov's notion of categorical crepant resolutions, other notions have also been introduced. Motivated by a conjecture of Bondal and Orlov, the idea behind these notions of resolutions is to find objects that behave like geometric resolutions would, even in absence of an underlying geometric object.

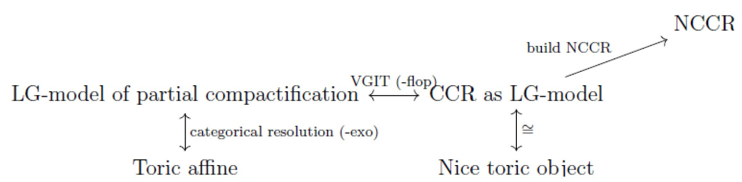
Non-commutative crepant resolutions

Let R be a normal noetherian domain.
A twisted non-commutative resolution of R is a reflexive Azumaya algebra Λ over R such that $\operatorname{gl}\dim \Lambda < \infty$. If Λ is trivial, then Λ is said to be a **non-commutative resolution** (NCR) of R. Assuming further R is Gorenstein, we call a twisted NCR **crepant** if it is additionally a Cohen-Macaulay R-module. If such a Λ is an NCR, then it is said to be a **non-commutative crepant resolution** (NCCR) of R.

Two conjectures

- Van den Bergh builds on the Conjecture of Bondal-Orlov and Kawamata and suggests that all resolutions, commutative or not, are derived equivalent.
- Furthermore, Van den Bergh proposes the following conjecture: An affine Gorenstein toric variety always has an NCCR.

Approaching the problem via exoflops



Thank you for your attention ☺