Gepner points in the Kähler moduli of K3 surfaces

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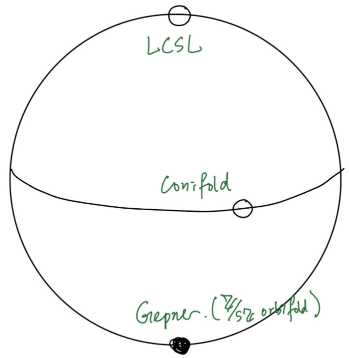
Example: $\{x_0^5 + x_1^5 + \cdots + x_4^5 + z \cdot x_0 x_1 \cdots x_4 = 0\} \subseteq \mathbb{CP}^4$.



Monodromy action on $H_3(X_z)$:

- LCSL $\longleftrightarrow$ maximally unipotent matrix
- Gepner $\longleftrightarrow$ matrix of order 5

Kontsevich: Should lift the monodromy to $\mathrm{Aut}(\mathrm{Fuk}(X_z))(\cong \mathrm{Aut} D^b(X^\vee))$

Horja:

- LCSL $\longleftrightarrow \otimes \mathcal{O}_{X^\vee}(1)$
- Conifold $\longleftrightarrow T_{\mathcal{O}_{X^\vee}}$ (Seidel–Thomas spherical twist)
- Gepner $\longleftrightarrow T_{\mathcal{O}_{X^\vee}} \circ (\otimes \mathcal{O}_{X^\vee}(1))$

The $\mathbb{Z}_5$-symmetry at the Gepner point is reflected on the fact that

$$\left(T_{\mathcal{O}_{X^\vee}} \circ (\otimes \mathcal{O}_{X^\vee}(1))\right)^5 = [2].$$

Given a Calabi–Yau manifold X, we would like to relate

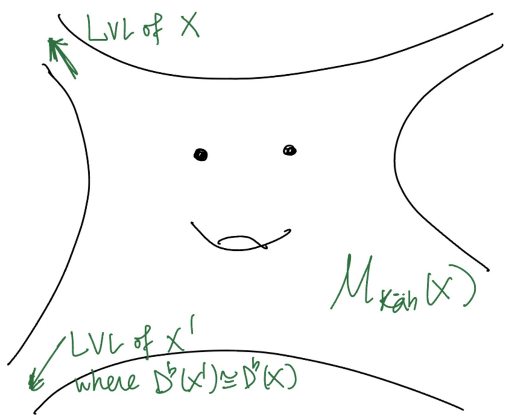
monodromy of "special points" in "$M_{\text{Käh}}(X)$"

with

elements in $\mathrm{Aut} D^b(X)$.

Remark: "$M_{\text{Käh}}(X)$" is not well-defined in general. However, it has a precise definition for $\dim(X) \leq 2$.

Bridgeland: $M_{\text{Käh}}(X)$ should be defined via stability conditions on $D^b(X)$.



Conjecture:

$$M_{\mathrm{cpx}}(X) \hookrightarrow \mathrm{Aut}\backslash\mathrm{Stab}(\mathrm{Fuk}(X))/\mathbb{C}$$
$$M_{\mathrm{Käh}}(X) \hookrightarrow \mathrm{Aut}\backslash\mathrm{Stab}(D^b(X))/\mathbb{C}$$

(Slogan: “deformation” and “stability” are mirror to each other.)

Conjecture: $\mathrm{Stab}(D^b(X))$ is contractible.

($\Longrightarrow \pi_1(M_{\mathrm{Käh}}(X))$ and $\mathrm{Aut}(D^b(X))$ should be related.)

$\sigma = (Z, P) \in \mathrm{Stab}(D)$ if:

- $Z \colon \mathrm{Ob}(D) \to \mathbb{C}$ such that $Z(B) = Z(A) + Z(C)$ for any exact triangle $A \to B \to C \to A[1]$ (central charge)
- $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ collection of full abelian subcategories of D (semistable objects of phase ϕ)

satisfying:

- $Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$ if $E \in P(\phi)$
- $P(\phi + 1) = P(\phi)[1]$
- $\mathrm{Hom}(P(\phi_1), P(\phi_2)) = 0$ if $\phi_1 > \phi_2$
- Harder–Narasimhan property
- Support property

Example: X = curve, and $\tau \in \mathbb{H}$

$$Z_\tau(E) = -\mathrm{deg}(E) + \tau \cdot \mathrm{rank}(E)$$

$$P(\phi) = \left\{ E \in \mathrm{Coh}(X) \text{ slope semistable with } Z_\tau(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi} \right\} (0 < \phi \leq 1)$$

Conjecture: $D = \mathrm{Fuk}(X, \omega)$, and $\Omega^{n,0}$ = holomorphic n-form

$$Z_\Omega(L) = \int_L \Omega; \qquad P_\Omega(\phi) = \{\text{sLag of phase } \phi \text{ w.r.t. } \Omega, \omega\}$$

Conjecture: $D = D^b(X)$, and ω ample
There exists a stability condition with $Z(E) \sim -\int_X e^{-(B+i\omega)}\mathrm{ch}(E)\hat{\Gamma}_X + \cdots$

Remark: $X =$ CY threefold. In the conjectural embedding

$$M_{\text{Käh}}(X) \hookrightarrow \mathrm{Aut}\backslash\mathrm{Stab}(D^b(X))/\mathbb{C}:$$

- The existence of stability condition is not known in general.
- $\dim(M_{\text{Käh}}(X)) = h^{1,1}$, while $\dim(\mathrm{Stab}(D^b(X))) = h^{0,0} + h^{1,1} + h^{2,2} + h^{3,3}$. It's highly nontrivial (probably not known) how to find $M_{\text{Käh}}(X)$ inside of $\mathrm{Aut}\backslash\mathrm{Stab}(D^b(X))/\mathbb{C}$.

Remark: $X =$ K3 surface. ($\dim(\mathrm{Stab}(D^b(X))) = h^{0,0} + h^{1,1} + h^{2,2}$) One can cut down 2 dimensions by:

- quotient the free $\mathbb{C}$-action on Stab
- impose the mirror condition of $\langle \Omega^{2,0}, \Omega^{2,0} \rangle = 0$

Bridgeland: $X = $ K3 surface. There is a covering

$$\mathrm{Stab}^\dagger D^b(X) \xrightarrow{\pi} P_0^+(X) \subseteq \mathrm{Hom}(N(X) := H^0 \oplus \mathsf{NS} \oplus H^4, \mathbb{C})$$

where

- $P(X) \cong \mathsf{GL}^+(2,\mathbb{R}) \cdot \{\exp(\beta + i\omega) | \beta, \omega \in \mathsf{NS}_\mathbb{R}, \omega^2 > 0\} \supseteq P^+(X)$
- $P_0^+(X) = P^+(X) \setminus \cup_{\delta \in N(X), \delta^2 = -2}\, \delta^\perp$

and $\mathsf{Deck}(\pi) \subseteq \mathrm{Aut}(D^b(X))$.

Conjecture:

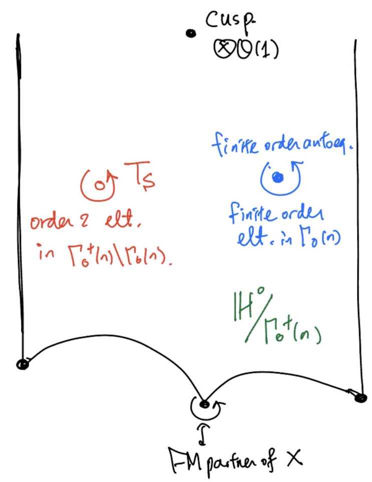
- $\mathrm{Aut}(D^b(X))$ preserves the component $\mathrm{Stab}^\dagger D^b(X)$
- $\pi_1(\mathrm{Stab}^\dagger D^b(X)) = \{1\}$

$\Longrightarrow$ very nice description of $\mathrm{Aut}(D^b(X)))$:

$$1 \to \pi_1(P_0^+(X)) \to \mathrm{Aut}(D^b(X)) \to \mathrm{Aut}^+ \widetilde{H}^*(X, \mathbb{Z}) \to 1$$

Bayer–Bridgeland: The conjecture holds for $\rho(X) = 1$. In fact, $\mathrm{Stab}^\dagger D^b(X)$ is contractible.

Theorem (F.–Lai): $X =$ K3 surface with $\rho(X) = 1$.
We give full classifications of finite subgroups of $\mathrm{Aut} D^b(X)$ and $\mathrm{Aut} D^b(X)/[1]$.



One of the key steps in the classification is to show that

any finite order element in $\mathrm{Aut}(D^b(X))/[1]$ fixes a point in $\mathrm{Stab}(D^b(X))/\mathbb{C}$.

Remark: This is a categorical analogue of the Nielsen realization problem of finite order elements in the mapping class groups of Riemann surfaces.

Idea: Consider the $\mathrm{Aut}(D)$-equivariant covering map

$$\mathrm{Stab}^{\dagger}_{\mathrm{red}}(D)/\mathbb{C} \xrightarrow{\pi} Q_0^+(D)$$

where $Q_0^+(D) = \{v \in \mathbb{P}(N(D) \otimes \mathbb{C}) \mid v^2 = 0, v\overline{v} > 0\} \setminus \bigcup_{\delta^2=-2} \delta^{\perp}$.

- It is not hard to show that finite order autoequivalence fixes a point in $Q^+(D)$ using basic Lie theory. One needs to show that the fixed point avoids $\delta^{\perp}$.
- Lift and get a fixed point in $\mathrm{Stab}^{\dagger}_{\mathrm{red}}(D)/\mathbb{C}$.

Suppose X is a K3 surface of $\rho(X) = 1$ and degree $2n$.

- We have $Q_0^+(D) \cong \mathbb{H} \setminus$ "(-2)-points".
- By Dolgachaev (1996) and Kawatani (2014), the action of $\mathrm{Aut}(D)$ on $Q_0^+(D)$ factors through $\mathrm{Im}(\mathrm{Aut}(D) \xrightarrow{f} \mathrm{PSL}(2,\mathbb{R})) = \Gamma_0^+(n)$ the Fricke modular group, where $\Gamma_0^+(n) = \left\langle \Gamma_0(n), \begin{bmatrix} & -1/\sqrt{n} \\ \sqrt{n} & \end{bmatrix} =: \omega_n \right\rangle$.

We showed that the following statements are equivalent:

- $f(\Phi)$ fixes a (-2)-point in $\mathbb{H}$
- $f(\Phi)$ is an involution, and $f(\Phi) = g_0\omega_n$ for some $g_0 \in \Gamma_0(n)$
- $\Phi = T_S\Psi$ for some spherical object S and some $\Psi \in \mathrm{Deck}(\pi)$

Moreover, we showed that autoequivalences of the form $T_S\Psi$ must be of infinite order in $\mathrm{Aut}(D)/[1]$. This resolves the first issue.

Let $\Phi \in \mathrm{Aut}(D)/[1]$ be of finite order, then the previous discussion shows that it fixes a point in $Q_0^+(D)$.

By combining:

- Kawatani (2019): $\pi_1(Q_0^+(D)) \cong \star_{\mathsf{free}} T_S^2$.
- Bayer–Bridgeland (2017): $\mathrm{Stab}^\dagger_{\mathsf{red}}(D)/\mathbb{C}$ is contractible.

we have $\mathrm{Deck}(\pi) \cong \star_{\mathsf{free}} T_S^2$.

We show that this is enough to imply that the fixed point of Φ in $Q_0^+(D)$ can be lifted to a fixed point in $\mathrm{Stab}^\dagger_{\mathsf{red}}(D)/\mathbb{C}$.

Other applications: For X K3 surface of Picard number 1, we show that

there exists an associated cubic fourfold

if and only if

there exists $\Phi \in \mathrm{Aut}(D^b(X))$ with $\Phi^3 = [2]$.

Categorical trichotomy: (up to $\mathrm{Deck}(\pi)$)

- $h_{\mathrm{cat}}(\Phi) > 0$
- $h_{\mathrm{cat}}(\Phi) = 0$ and $h_{\mathrm{poly}}(\Phi) > 0$:
  - $h_{\mathrm{poly}}(\Phi) = 1$: spherical twist
  - $h_{\mathrm{poly}}(\Phi) = 2$: monodromy around the cusps
- $h_{\mathrm{cat}}(\Phi) = h_{\mathrm{poly}}(\Phi) = 0$: finite order autoequivalences.

Thank you for your attention!