The Eisenstein spectrum

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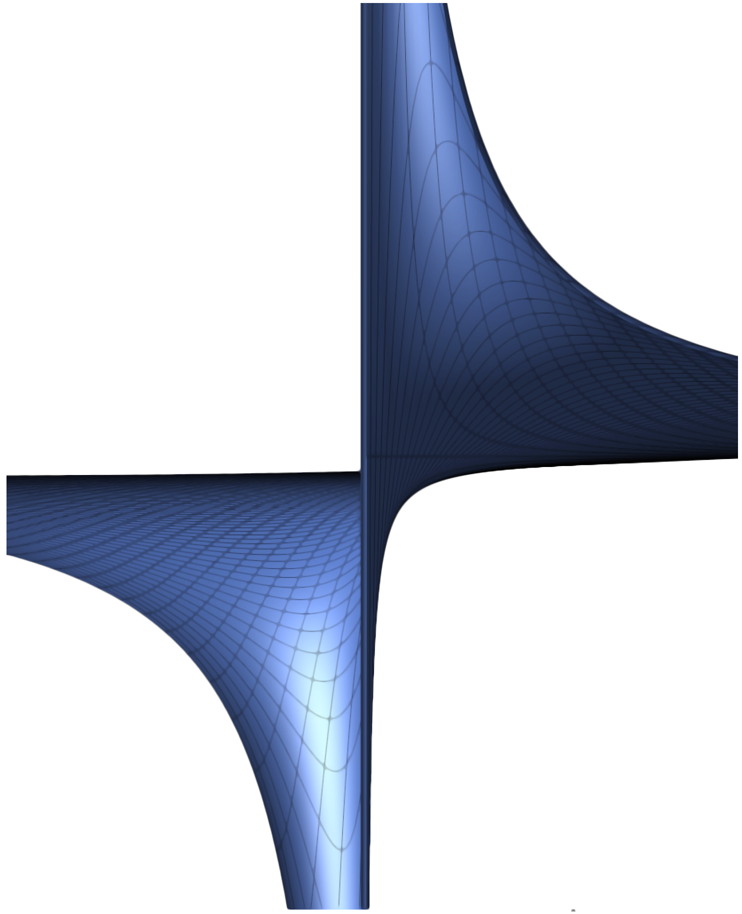
based on joint work with David Kazhdan

In this talk, we will *see* 3 *a priori* different kind of poles, having to do with:

- *resolvents and spectra of operators*,
- ζ-functions *and* Eisenstein series,
- *the pole in*

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \ldots ,$$

the geometry behind which we will discuss in due course.

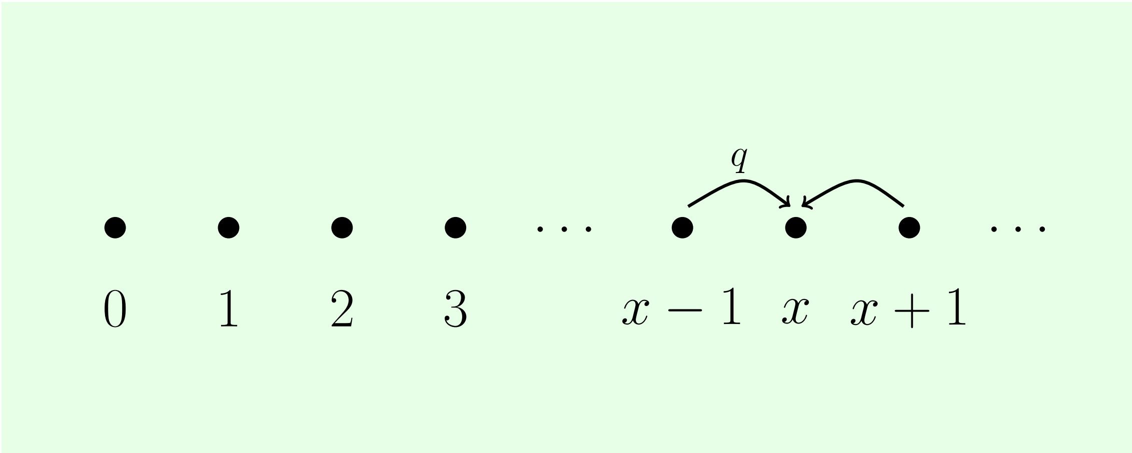


① Spectra, resolvents, wave packets, . . .

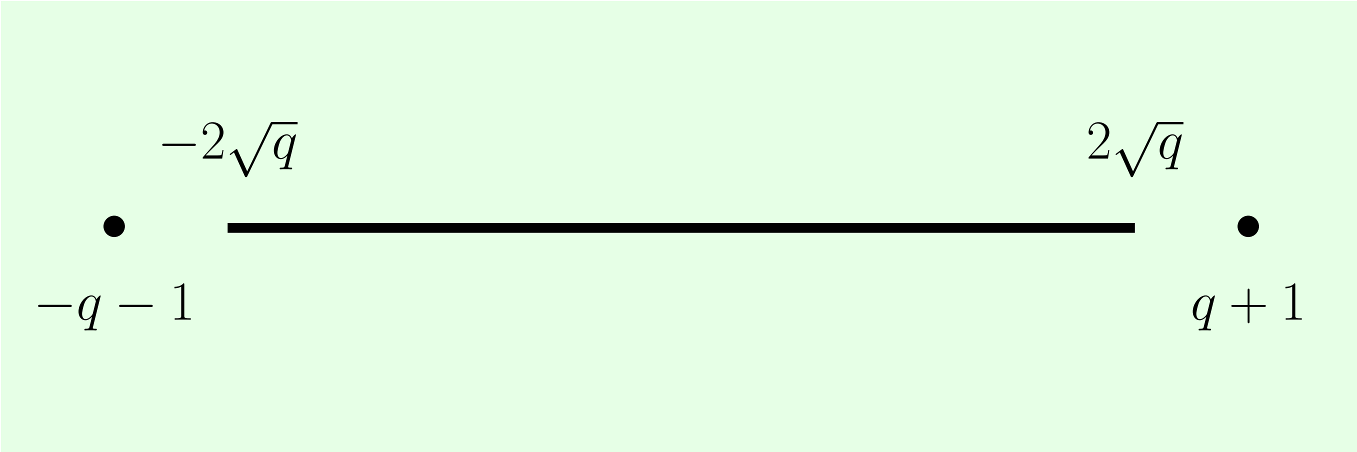
For a very, very, very basic *example*, take the difference operator

$$[\Delta f](x) = qf(x-1) + f(x+1), \quad f(-1) = f(1),$$

where $x \in \{0, 1, 2, \dots\}$ and $q > 1$.



It has *eigenvalues* $\pm(q+1)$ *and a* continuous spectrum filling $[-2\sqrt{q}, 2\sqrt{q}]$.



This means the resolvent $(\Delta - z)^{-1}$ has *a* pole *at* $z = \pm(q+1)$, *a* jump *across* $[-2\sqrt{q}, 2\sqrt{q}]$, *and* holomorphic *elsewhere*.

Stone's formula gives the spectral projectors in the spectral theorem in terms of the residues/jumps of $(\Delta - z)^{-1}$.

How can one read this information off the behavior of nonnormalizable eigenfunctions of Δ, also know as Jost functions *etc.* ?

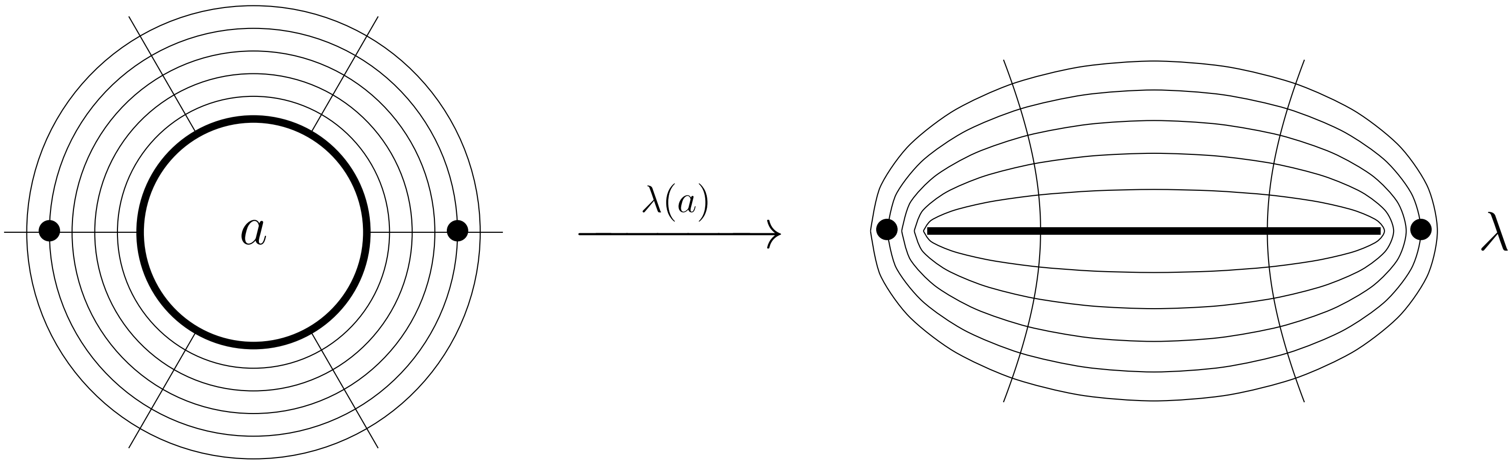
It is *easy* to construct *eigenfunctions* such that

$$E(x,a) \sim (q^{1/2}a)^x\,, \quad x \to \infty\,, \quad |a| \gg 1\,,$$

with *eigenvalue*

$$\Delta E(x,a) = \lambda(a) E(x,a)\,, \quad \lambda(a) = q^{1/2}(a + a^{-1})\,.$$

The function $\lambda(a) = q^{1/2}(a + a^{-1})$ is the one of the Zhukovsky *airfoil fame*:



It takes $\{|a| > 1\}$ to $\mathbb{C} \setminus [-2\sqrt{q}, 2\sqrt{q}]$.

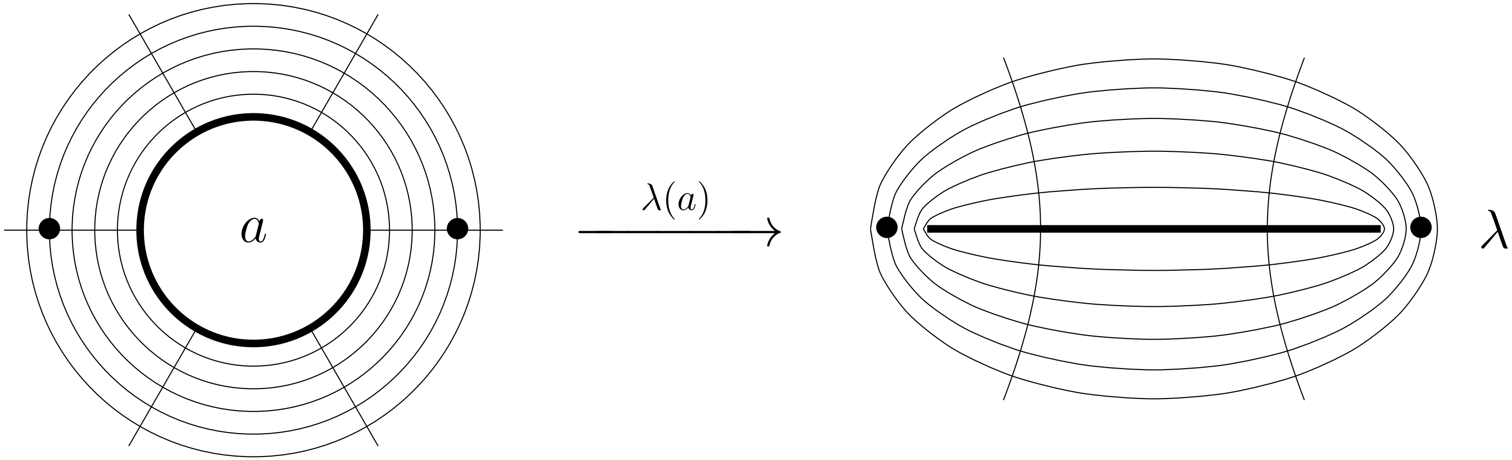
The points $a = \pm q^{1/2}$ go to the *eigenvalues* of Δ.

While $E(x,a) \notin L^2$, the corresponding wave packets

$$E_\omega(x) = \oint_{|a|=c\gg 1} \omega(a) E(x,a) \, \frac{da}{2\pi i a} \, ,$$

with $\omega(a) \in \mathbb{C}[a^{\pm 1}]$, are dense in L^2 and satisfy

$$(\Delta - z)^{-1} E_\omega(x) = \oint_{|z|>|a|=c\gg 1} \frac{\omega(a)}{\lambda(a) - z} \, E(x,a) \, \frac{da}{2\pi i a} \, .$$



From the formula for $(\Delta - z)^{-1}E_\omega$, we see that:

- the function $E(x,a)$ must have poles at $a = \pm q^{1/2}$, with residue the corresponding eigenfunction of Δ,
- the 2:1 map $\{|a|=1\} \xrightarrow{\lambda} [-2q^{1/2}, 2q^{1/2}]$ creates the jump in the resolvent.

Logically, it is *easier* to invert this *reasoning*, *and* identify L^2, *and* the op-*erator* Δ, with the completion of the Laurent polynomials $\omega(a)$

$$\begin{array}{ccc} \mathbb{C}[a^{\pm 1}] & \xrightarrow{\omega \mapsto E_\omega} & L^2 \\ \big\downarrow \lambda(a) & & \big\downarrow \Delta \\ \mathbb{C}[a^{\pm 1}] & \xrightarrow{\omega \mapsto E_\omega} & L^2 \end{array}$$

with *respect* to the seminorm

$$\begin{aligned} \|E_\omega\|^2 &= \int_{|a|=c\gg 1} (\dots) \\ &= \int_{|a|=1} (\dots) + \underbrace{*|\omega(q^{1/2})|^2 + *|\omega(-q^{1/2})|^2}_{\text{residues from the poles of } E(x,a)} \ . \end{aligned}$$

② ζ-functions, Eisenstein series, ...

The Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{primes } p} \frac{1}{1 - p^{-s}}$$

converges for $\Re s > 1$ and has a single pole at $s = 1$.

The more symmetric version

$$\xi(s) = \pi^{-s/2} \Gamma(s/2)\, \zeta(s)\,, \quad \xi(1-s) = \xi(s)$$

has poles at $s \in \{0, 1\}$.

Among the many, many generalizations of $\zeta(s)$, we will meet:

- the functions $\zeta_{\mathbb{F}}(s), \xi_{\mathbb{F}}(s)$ for a global field $\mathbb{F}$,
- the Eisenstein series.

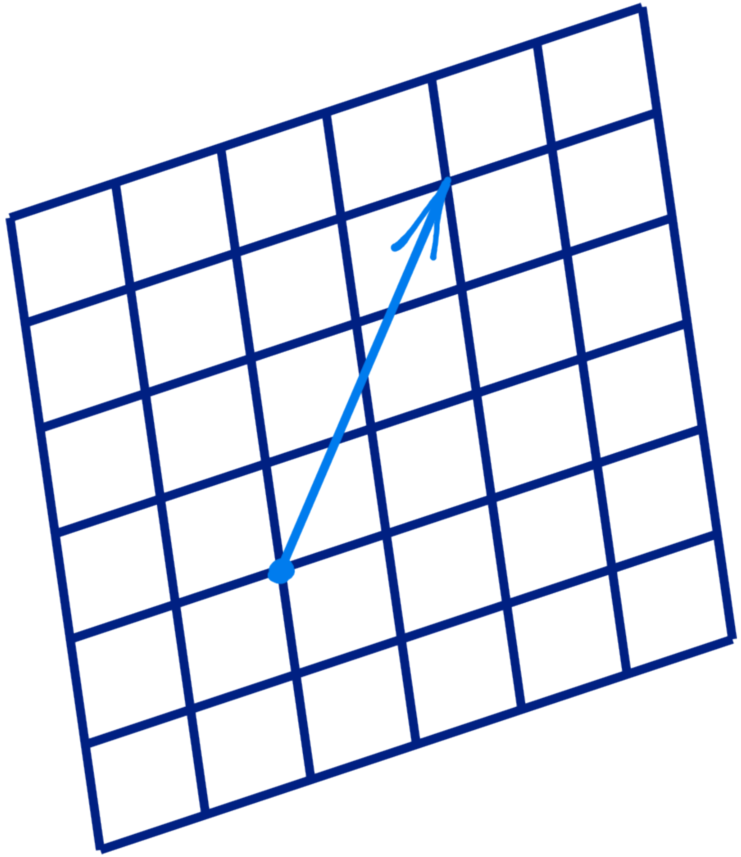
Let $\Gamma \subset \mathbb{R}^2$ be a unimodular lattice.

The classical Eisenstein series for $\mathsf{G} = SL(2, \mathbb{Q})$ is given by

$$\mathsf{Eis}(\Gamma, s) = \frac{1}{2} \sum_{\text{primitive } \gamma \in \Gamma} \frac{1}{\|\gamma\|^{s+1}}$$

for $\Re s > 0$, and by analytic continuation for other values of s, with a pole at $s = 1$.

For $\Gamma \subset \mathbb{R}^1$, dropping the primitive condition, this would be $\zeta(s+1)$.

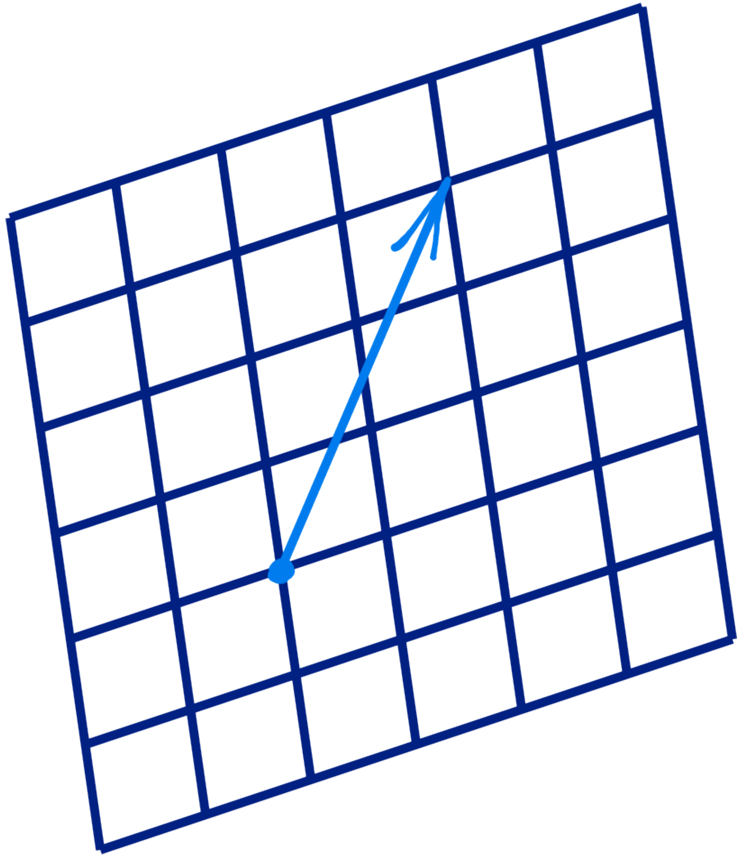


Primitive vectors γ in Γ modulo ± 1, *are* indexed by their slope, which takes values in $\mathbb{Q} \cup \{\infty\}$. Equivalently,

$$\mathbb{Z}\gamma \in \mathbb{P}^1(\mathbb{Q}) = SL(2,\mathbb{Q})/\text{Borel subroup}\,.$$

For *any* projective homogeneous variety G/P over *a* global field $\mathbb{F}$ one can count points of $\mathsf{G}/\mathsf{P}(\mathbb{F})$ *according* to their <u>height</u>.

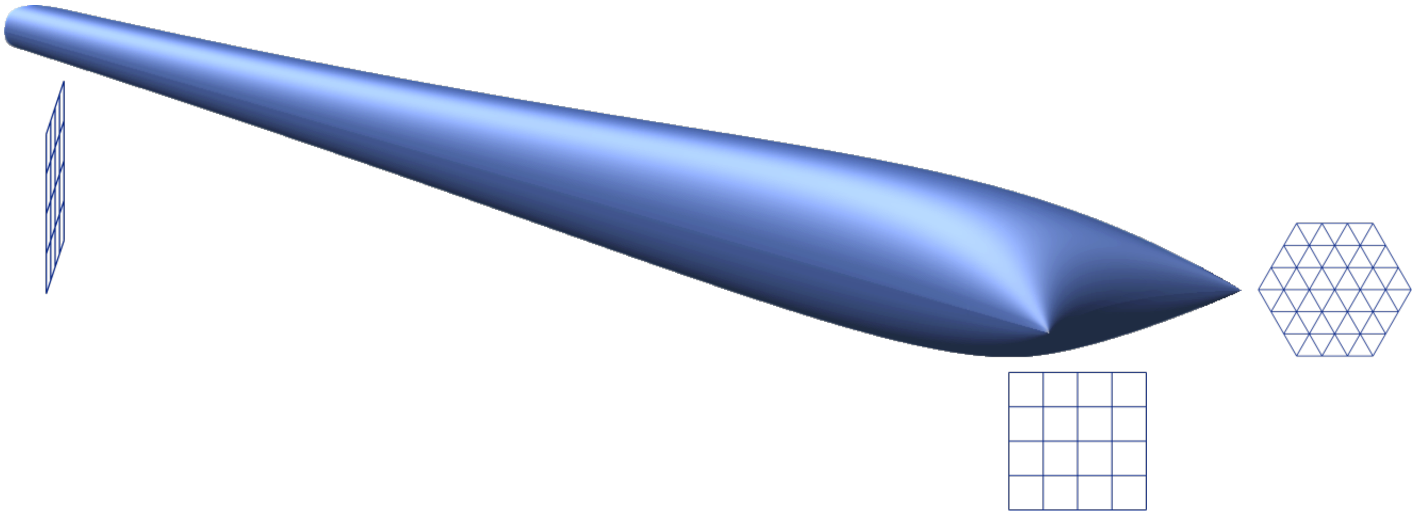
These height ζ-functions, *a*.k.*a*. Eisenstein series, have parallel *analytic* properties.



As a function of the lattice

$$\Gamma \in SO(2,\mathbb{R})\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})\,,$$

$\mathrm{Eis}(\Gamma, s)$ is an eigenfunction of the Laplace operator and Hecke operators.



It is not in L^2 for any value of s, just like $E(x,a)$ from before. In fact, $E(x,a)$ are Eisenstein series for $\mathsf{G} = PGL(2, \mathbb{F}_q(t))$.

The problem studied by Langlands and many people since, including Mœglin, Waldspurger, Heiermann, de Martino, and Opdam, is to determine the spectrum of the Laplace and Hecke operators in the intersection of L^2 and the span of the Eisenstein series.

Equivalently, one can talk about the corresponding wave packets

$$\begin{aligned}\mathrm{Eis}_f &= \int_{\Re s = c \gg 0} f(s)\,\mathrm{Eis}(\Gamma, s)\,ds \\ &= \frac{1}{2} \sum_{\text{primitive } \gamma \in \Gamma} f^\vee(\|\gamma\|)\,,\end{aligned}$$

known as the pseudo-Eisenstein series[1].

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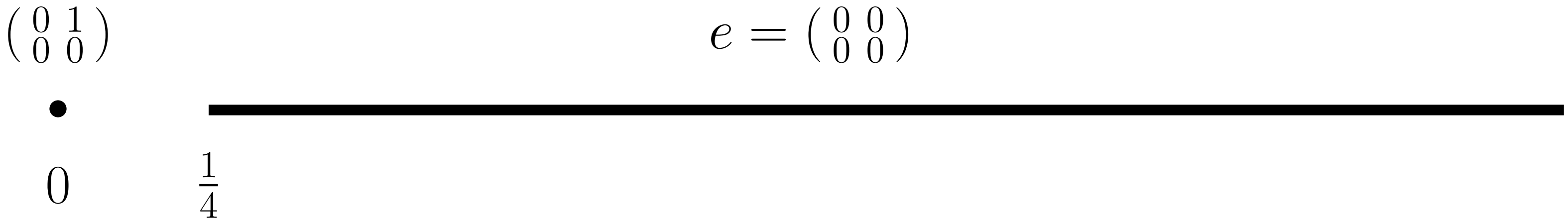
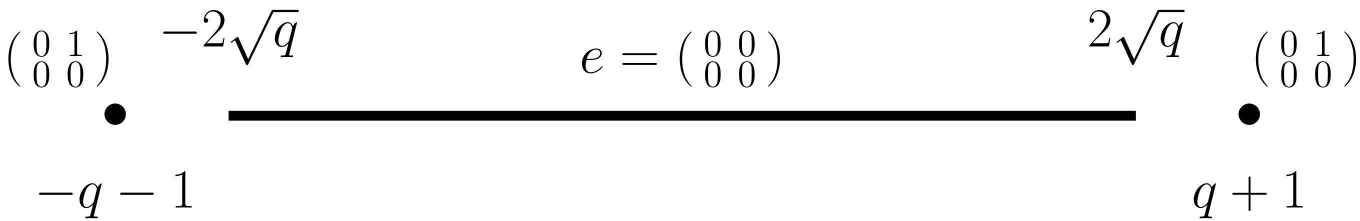
[1]Here $f^\vee(x) = \int x^{-s-1} f(s) ds$ is the Mellin-Fourier-Laplace dual of f and we can assume $f^\vee$ to be C^∞ and with compact support. People call such functions f the Paley-Wiener functions.

Challenging conjectures (due to Langlangs, Arthur, ...), some of which were already proven, link this spectrum to nilpotent elements e in the Langlands dual Lie algebra. Incidentally, for $SL(2,\mathbb{Q})$, this means



which is the Lie algebra $q \to 1$ version of the group-theory picture



from earlier.

Concretely, in part ①, we have

$$\lambda(a) = q^{1/2}\,\mathrm{tr}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$$

and the spectrum is described by

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in q^{h/2}\,SU(2)^e \subset SL(2,\mathbb{C})\,.$$

Here $SU(2)^e$ means the centralizer of e and h solves $[h,e] = 2e$, that is, h is the middle element of the $\mathfrak{sl}_2$-triple $\langle f, h, e\rangle$ determined by e.

For $e = 0$, we have $h = 0$, and hence we get $|a| = 1$. For $e \neq 0$, we have $h = \mathrm{diag}(1, -1)$, $SU^e = \{\pm 1\}$, giving $a = \pm q^{1/2}$.

These conjectures been approached through residue calculus in certain integrals first written down by Langlands (who himself carried out spectacularly complex computations for groups of type G_2).

In those computations, nearly everything cancels out, except one unexpected term that was linked to the subregular nilpotent in $G_2^\vee = G_2$, ...

Concretely, for $\mathsf{G}/\mathsf{B}(\mathbb{F})$ one has to push the contour in

$$(\mathsf{Eis}_{f_1}, \mathsf{Eis}_{f_2})_{L^2} = \int_{\Re\lambda=\lambda_0\gg 0} d\lambda \sum_{w\in W} f_1(\lambda)\overline{f_2}(-w\lambda) \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{\xi_{\mathbb{F}}(\alpha(\lambda))}{\xi_{\mathbb{F}}(\alpha(\lambda)+1)}$$

towards $\Re\lambda = 0$, where

$$\begin{aligned} \lambda &= \text{a vector in } \mathbb{C}^{\operatorname{rk}\mathsf{G}} = \text{Lie algebra of max torus in } \boxed{\mathsf{G}^\vee} \\ W &= \text{a finite reflection group acting on } \mathbb{C}^{\operatorname{rk}\mathsf{G}} \\ \{\alpha\} &= \text{ the roots } (\approx \text{ equations of reflection hyperplanes}) \\ \xi_{\mathbb{F}} &= \text{ the completed } \zeta\text{-function of } \mathbb{F} \end{aligned}$$

For $\mathsf{G} = E_8$, we have

$$\mathsf{rk} = 8\,, \quad |W| = 696729600\,, \quad |\{\alpha > 0\}| = 120\,.$$

While $\sum_{i=0}^{8} \binom{120}{i} \approx 10^{12}$, there are only 70 nilpotent conjugacy classes, and *each* contributes an *easily describable* piece to the spectrum.

Inverting the logic like we did in the baby example, we can describe the Hilbert space in questions as the completion of (Paley-Wiener) functions[2]

$$\begin{array}{ccc} \mathrm{Fun}(\lambda) & \xrightarrow{f \mapsto \mathrm{Eis}_f} & L^2 \\ {\scriptstyle \text{mult by } \mathbb{C}[\lambda]^W} \downarrow & & \downarrow {\scriptstyle \text{Laplace and Hecke}} \\ \mathrm{Fun}(\lambda) & \xrightarrow{f \mapsto \mathrm{Eis}_f} & L^2 \end{array}$$

with respect to the seminorm

$$\|f\|^2_{\mathrm{Eis}} = \|\mathrm{Eis}_f\|^2_{L^2} \,.$$

[2] Mellin-Fourier-Laplace transforms of C^∞-functions with compact support, as before

To see the expected connection to nilpotent elements $e \in \operatorname{Lie} \mathsf{G}^\vee$ is the Lie algebra of the Langlands dual complex group, we must have

$$\|f\|^2_{\mathsf{Eis}} = \int_{\Re\lambda=\lambda_0\gg 0} \sum_{w\in W} \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{\xi}{\xi} \cdots$$
$$\stackrel{!}{=} \sum_e \int_{\frac{1}{2}h+\mathsf{Lie}(\mathsf{K}^\vee)^e} |\Pi f|^2 d\mu_{\mathsf{spectral}} \,.$$

Here $f \to \Pi f$ is a certain projection to W-invariant functions of λ, that is, conjugation-invariant functions on $\operatorname{Lie} \mathsf{G}^\vee$, and $d\mu_{\mathsf{spectral}}$ is a measure in the Lebesgue measure class.

Geometrically, it may be slightly easier to work with a bilinear form

$$\mathsf{Fun}(\lambda) \otimes_{\mathsf{Fun}(\lambda)^W} \mathsf{Fun}(\lambda) \ni f_1 \otimes f_2 \to (f_1, \overline{f_2})_{\mathsf{Eis}} \,.$$

We want to write this bilinear form as a sum over nilpotents in $\mathrm{Lie}\, \mathsf{G}^\vee$.

③ Characters as distributions

The geometric series

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \dots\,,$$

may be interpreted as the trace of the element $a \in GL(1)$ acting by

$$x \xrightarrow{\ a\ } ax$$

on

$$\begin{aligned} \mathbb{C}[x] &= \bigoplus_{k=0}^{\infty} \mathbb{C}\, x^k\,, \\ &= \text{functions on a line } \mathsf{X} = \mathsf{A}^1 \text{ with coordinate } x\,. \end{aligned}$$

We note that

$$\text{poles of } \frac{1}{1-a} = \left\{ a \,\middle|\, \mathsf{X}^a \text{ is not compact} \right\}\,.$$

Moreover,

$$\begin{aligned}\text{multiplicity of } a^k \text{ in } \mathbb{C}[\mathsf{A}^1] &= \delta_{k\geq 0}\\ &= \int_{|a|=c<1} a^{-k}\,\frac{1}{1-a}\,d_{\mathsf{Haar}}a\end{aligned}$$

It is thus natural to interpret

$$\left(\mathrm{tr}_{\mathbb{C}[x]}\, a\right)\, d_{\mathsf{Haar}}a\,,$$

with a choice of the integration contour, as a holomorphic distribution on $GL(1)$ or ordinary distribution on $U(1)$.

This *can be generalized as* follows:

For many noncompact *algebraic varieties* X, irr*e*ducible *represe*ntations of Aut(X) *appear* in $\mathbb{C}[X]$ with finit*e* multiplicity, *and hence* th*e* charact*er* of $\mathbb{C}[X]$ is well-d*e*fin*ed as a* conjugation-invariant distribution on Aut(X).

This can be further generalized in two ways:

- replace functions $\mathscr{O}_{\mathsf{X}}$ on X by finitely generated modules over functions, that is, by coherent sheaves $\mathscr{F}$ on X. We assume $\mathscr{F}$ equivariant for some $\mathsf{G} \subset \mathrm{Aut}(\mathsf{X})$.
- replace globally defined functions, or global sections of $\mathscr{F}$, by other cohomology groups $H^i(\mathsf{X}, \mathscr{F})$.

Particularly important are Euler characteristics

$$\chi(\mathsf{X}, \mathscr{F}) = \bigoplus (-1)^i H^i(\mathsf{X}, \mathscr{F}) \in \text{virtual representations of } \mathsf{G}$$

Euler characteristics $\chi(\mathsf{X}, \mathscr{F})$ are additive in both $\mathscr{F}$ and X. Namely,

$$\chi(\mathscr{F}) = \chi(\mathscr{F}_1) + \chi(\mathscr{F}_2)\,,$$

for $0 \to \mathscr{F}_1 \to \mathscr{F} \to \mathscr{F}_2 \to 0$ and also

$$\chi(\mathsf{X}, \cdot\,) = \chi(\mathsf{X} \setminus \mathsf{Y}, \cdot\,) + \chi(\mathsf{Thom}(\mathsf{Y} \to \mathsf{X}), \cdot\,)$$

for $\mathsf{Y} \subset \mathsf{X}$ closed. Here

$$H^i(\mathsf{Thom}(\mathsf{Y} \to \mathsf{X}), \cdot\,) = H^i_{\mathsf{Y}}(\mathsf{X}, \cdot\,)$$

is just a fancy but suggestive notation for local cohomology.

This means that in our computations descend to $[\mathscr{F}] \in K_{\mathsf{G}}(X)$ and to the class $[\mathsf{X}]$ in a suitable scissors congruence group of varieties.

Recall from part ② that we are looking for a linear functional on

$$\mathsf{Fun}(\lambda) \otimes_{\mathsf{Fun}(\lambda)^W} \mathsf{Fun}(\lambda)$$

where

$$\mathsf{Fun}(\lambda) = \begin{cases} \mathbb{C}[\mathsf{A}^\vee] \text{ for a function field } \mathbb{F}, \\ \text{PW functions on } \mathrm{Lie}\,\mathsf{A}^\vee \text{ for a number field } \mathbb{F}. \end{cases}$$

In topology, this has to do with the K-theory/cohomology of the flag variety for $\mathsf{G}^\vee$ and, in particular for

$$\mathsf{X} = T^*(\mathsf{B}^\vee \backslash \mathsf{G}^\vee / \mathsf{B}^\vee)$$

we have

$$K(\mathsf{X}) = \mathbb{Z}[\mathsf{A}^\vee] \otimes_{\mathbb{Z}[\mathsf{A}^\vee]^W} \mathbb{Z}[\mathsf{A}^\vee]\,.$$

In fact,

$$(f_1, f_2)_{\mathsf{Eis}} = \chi\left(\mathsf{X}, (f_1 \boxtimes \bar{f}_2) \otimes \dots\right) .$$

where "..." stand for some fixed K-theory class, the exact form of which is of little importance for what follows.

But, if one wants to be more precise, we have[3]

$$(f_1, f_2)_{\mathsf{Eis}} = \chi\left(\mathsf{X}, (f_1 \boxtimes \bar{f}_2) \otimes \mathsf{L\text{-}genus}(\mathsf{X})\right) \,.$$

Here the L-genus refers to a certain Galois-equivariant genus, the full discussion of which will be left out.

In principle, L-genus has to do with actions of groups of the form

$$1 \to \Gamma' \to \Gamma \to \mathsf{Norms}(\mathbb{A}_{\mathbb{F}}^{\times}) \to 1\,.$$

In the case at hand, Γ acts via $\mathsf{Norms} \subset \mathbb{R}_{>0}$ scaling of the cotangent fibers of X and the L-genus specializes to the genus defined by the function $\xi_{\mathbb{F}}$.

[3] This follows from definitions and equivariant localization. Deeper meaning may be discussed separately.

Famously, there is the Springer map

$$\pi : T^*(\mathsf{G}^\vee/\mathsf{B}^\vee) \to \mathsf{Nilp}(\mathsf{Lie}(\mathsf{G}^\vee)$$

which is the resolution of the singularities of the nilpotent cone.

This descends to a map

$$\bar{\pi} : \mathsf{X} = T^*(\mathsf{B}^\vee\backslash\mathsf{G}^\vee/\mathsf{B}^\vee) \to \mathscr{N} = \mathsf{Nilp}(\mathsf{Lie}(\mathsf{G}^\vee)/\mathsf{G}^\vee\,.$$

The target here has finitely many orbits, and

$$\bar{\pi}^{-1}(e) = \pi^{-1}(e) \times \pi^{-1}(e)\,.$$

From the diagram

$$\begin{CD} \mathsf{X} @<<< \pi^{-1}(e) \times \pi^{-1}(e) \\ @V{\bar{\pi}}VV @VVV \\ \mathscr{N} @<<< \mathrm{pt}/(\mathsf{G}^\vee)^e \end{CD}$$

we compute $(f_1, \overline{f}_2)_{\mathsf{Eis}} = \chi(\mathsf{X}, \dots)$ is a sum over the nilpotents e.

Each term, by orthogonality of characters, is an integral over the maximal compact subgroup in $(\mathsf{G}^\vee)^e$.

The projector $f \to \Pi f$ comes from the integration over $\pi^{-1}(e)$. The spectral measure comes from various normal bundles including the unipotent radical of $(\mathsf{G}^\vee)^e$.

This concludes the proof and the talk.