Twisted tensor product, smooth DG algebras, and noncommutative resolutions of singular curves

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Conference on Algebraic Geometry and Mathematical Physics
HSE/BIMSA
7th November, 2024, Moscow

Let X be a q-comp and q-sep scheme. With any X we can associate tri category of perfect complexes $\mathrm{perf}{-}X$ and derived category $\mathcal{D}_{\mathrm{Qcoh}}(\mathcal{O}_X{-}\mathrm{Mod})$. The category $\mathrm{perf}{-}X$ is equi to the subcategory of comp objects in $\mathcal{D}_{\mathrm{Qcoh}}(\mathcal{O}_X{-}\mathrm{Mod})$.

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If X is a q-comp and q-sep scheme, then $\mathscr{P}erf\!-X$ is q-equi to $\mathscr{P}erf\!-\mathscr{R}$ for a coh-ly bounded DGA $\mathscr{R}$ (A.Neeman; A.Bondal & M.Van den Bergh).

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There are several natural ways to obtain smooth proper DN schemes. One is deformations of commutative varieties. For example, Sklyanin algebras. This method works well for Fano varieties. Another way is to consider admissible subcategories $\mathcal{N} \subset \mathrm{perf}\text{–}X$, where X is smooth projective.

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Example: **[Noncommutative K3]** 1) Let $X \subset \mathbb{P}^5$ be a 4–dim cubic. Then $\mathrm{perf}\text{–}X = \langle \mathcal{O}_X(-2), \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{N} \rangle$. The subcategory $\mathcal{N}$ is a noncommutative K3. We have 20–dimensional family of NC K3.

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2) Another example $X = Gr(2,5) \cap H \cap Q$. Then $\mathrm{perf}\text{–}X$ has a decomposition $\langle E(-1), \mathcal{O}_X(-1), E, \mathcal{O}_X, \mathcal{M} \rangle$. Subcategories $\mathcal{M}$ provide another 20–dimensional family of NC K3.

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Conjecturally, there are many different 20–dimensional families of NC K3.

They are related to even positive forms $\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}$, $4ab - c^2 > 0$.

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It is a mirror to a rigid 3–dim CY with $h^{1,1}(Y) = 84$ and $h^{1,2}(Y) = 0$ that is a quotient $E^3/(\mathbb{Z}/3)^2$, where E is Fermat cubic $x^3 + y^3 + z^3 = 0$, and the group acts on (z_1, z_2, z_3) as (ρ^a, ρ^b, ρ^c) and $a + b + c \equiv 0(3)$.

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If $\pi : \widetilde{\mathrm{Y}} \to \mathrm{Y}$ is a usual resolution of singularities of a singular variety Y that has rational singularities, then the functor $\mathbf{L}\pi^* : \mathrm{perf}\text{–}\mathrm{Y} \to \mathrm{perf}\text{–}\widetilde{\mathrm{Y}}$ delivers an example of a smooth resolution of Y.

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It can be extended to the case of geometric smooth proper DN schemes. A gluing $\mathscr{N}_1 \underset{\mathsf{T}}{\oplus} \mathscr{N}_2$ of any such DN schemes via a perfect T is geometric.

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The class of DN schemes $\mathscr{X} = \mathscr{P}erf\text{–}\mathscr{R}$ with f.d. DGA $\mathscr{R}$ contains all phantoms, i.e. smooth proper $\mathscr{X}$ with trivial K-theory. (A. Efimov).

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There is a direct way to describe twisted tensor products. Let $\phi : \mathscr{A} \otimes_{\mathscr{R}} \mathscr{B} \xrightarrow{\sim} C$ be the canonical isomorphism used in the definition of the twisted tensor product. Then, we can define a map

$$\tau : \mathscr{B} \otimes_{\mathscr{R}} \mathscr{A} \to \mathscr{A} \otimes_{\mathscr{R}} \mathscr{B} \quad \text{by the rule} \quad \tau(b \otimes a) := \phi^{-1}(i_B(b) \cdot i_A(a)).$$

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An $\mathscr{R}$–bilinear map τ is called a **twisting map** and we denote the DG $\mathscr{R}$–ring $\mathscr{C}$ by $\mathscr{A} \otimes^{\tau}_{\mathscr{R}} \mathscr{B}$.

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In such a case we have a special twisting map $\mathbf{v} : \mathscr{B} \otimes_{\mathscr{R}} \mathscr{A} \to \mathscr{A} \otimes_{\mathscr{R}} \mathscr{B}$:

$$\mathbf{v}(b \otimes a) = \epsilon_{\mathscr{A}}(\pi_{\mathscr{B}}(b)) \cdot a \otimes 1 + 1 \otimes b \cdot \epsilon_{\mathscr{B}}(\pi_{\mathscr{A}}(a)) - \epsilon_{\mathscr{A}}(\pi_{\mathscr{B}}(b)) \otimes \epsilon_{\mathscr{B}}(\pi_{\mathscr{A}}(a)).$$

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Theorem: *Let $\mathscr{R}$ be a f.d. DGA. Let $\mathscr{A}$ and $\mathscr{B}$ be f.d. $\mathscr{R}$–rings such that $\mathscr{A}$ has an augmentation $\pi_A : \mathscr{A} \to \mathscr{R}$ with an ideal $\mathsf{I}_{\mathscr{A}} = \operatorname{Ker} \pi_{\mathscr{A}}$ and $\mathscr{B}$ is semi-projective as a left DG $\mathscr{R}$–module.*
Let $\mathscr{C} = \mathscr{A} \otimes^{\tau}_{\mathscr{R}} \mathscr{B}$ be a twisted tensor product. Assume that $\mathsf{I}_{\mathscr{A}} \otimes_{\mathscr{R}} \mathscr{B} \subset \mathscr{C}$ is a DG ideal which is contained in the ext radical $(\mathsf{J}_{\mathscr{C}})_+$.
If $\mathscr{A}, \mathscr{B}$ are smooth, then the DGA $\mathscr{C} = \mathscr{A} \otimes^{\tau}_{\mathscr{R}} \mathscr{B}$ is also smooth.

Suppose $\mathscr{A}$ and $\mathscr{B}$ have $\mathscr{R}$–augmentations $\pi_{\mathscr{A}} : \mathscr{A} \to \mathscr{R}, \pi_{\mathscr{B}} : \mathscr{B} \to \mathscr{R}$ for which compositions $\pi_{\mathscr{A}} \circ \epsilon_{\mathscr{A}}$ and $\pi_{\mathscr{B}} \circ \epsilon_{\mathscr{B}}$ are the identity maps.
In such a case we have a special twisting map $\mathbf{v} : \mathscr{B} \otimes_{\mathscr{R}} \mathscr{A} \to \mathscr{A} \otimes_{\mathscr{R}} \mathscr{B}$:

$$\mathbf{v}(b \otimes a) = \epsilon_{\mathscr{A}}(\pi_{\mathscr{B}}(b)) \cdot a \otimes 1 + 1 \otimes b \cdot \epsilon_{\mathscr{B}}(\pi_{\mathscr{A}}(a)) - \epsilon_{\mathscr{A}}(\pi_{\mathscr{B}}(b)) \otimes \epsilon_{\mathscr{B}}(\pi_{\mathscr{A}}(a)).$$

Thus, we obtain DGA $\mathscr{A} \otimes^{\mathbf{v}}_{\mathscr{R}} \mathscr{B}$ for which $(1 \otimes b)(a \otimes 1) = \mathbf{v}(b \otimes a) = 0$, whenever $a \in \mathsf{I}_{\mathscr{A}}, b \in \mathsf{I}_{\mathscr{B}}$, where $\mathsf{I}_{\mathscr{A}} = \operatorname{Ker} \pi_{\mathscr{A}}$ and $\mathsf{I}_{\mathscr{B}} = \operatorname{Ker} \pi_{\mathscr{B}}$.

Let $Q_{n,m}$ be a quiver with two vertices $\mathbf{1}, \mathbf{2}$ and with n arrows $\{c_1, \ldots, c_n\}$ from $\mathbf{1}$ to $\mathbf{2}$ and m arrows $\{b_1, \ldots, b_m\}$ from $\mathbf{2}$ to $\mathbf{1}$.

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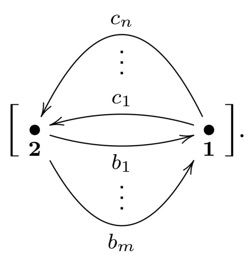

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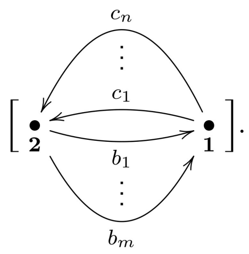

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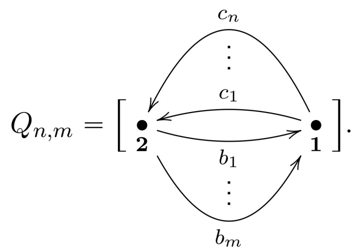

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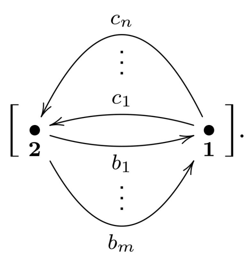
$Q_{n,m} =$ 

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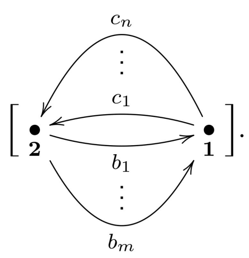
$Q_{n,m} =$ 

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We describe new families of smooth algebras with two simple modules. We start with a quiver of the form $Q_{n,m}$ and fix a sequence of integers $\bar{\mathfrak{k}} = \{k_1, \cdots, k_m\}$ such that $0 \leqslant k_m \leqslant \cdots \leqslant k_1 \leqslant n$.

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1) $b_i c b_j$ for any $c \in C$, when $i \leqslant j$;
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In the case $\mathfrak{F} = \varnothing$, the algebra $R_{\varnothing}$ is isomorphic to $K_n = \Bbbk Q_{n,0}$.

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The algebra $R_{\mathfrak{G}}$ admits many different augmentations $R_{\mathfrak{G}} \to K(V_m)$. They are related to specifying a complement to V_m in C. We fix the augmentation which is related to the vector subspace $W_m \subseteq C$. Such augmentation $\pi : R_{\mathfrak{G}} \to K(V_m)$ is given by setting $\pi(W_m) = 0$.

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This theorem is directly extended to the case of DGAs. We have

$$\mathscr{R}_{\mathfrak{F}}(\chi) \cong (K(V_m)(\chi(\varepsilon_0)) \otimes_S^{\mathbf{v}} K_1^\circ(\chi(\varepsilon_m))) \otimes_{K(V_m)(\chi(\varepsilon_0))}^{\mathbf{v}} \mathscr{R}_{\mathfrak{G}}(\chi').$$

Theorem: *For any family $\mathfrak{F}$ which satisfies property (G), the algebra $R_{\mathfrak{F}}$ has finite global dimension and any DG algebra $\mathscr{R}_{\mathfrak{F}}(\chi)$ is smooth.*

The K-theory of the categories $\mathrm{perf}\text{–}\mathscr{R}_{\mathfrak{F}}(\chi_0)$ is equal to $K_*(\Bbbk) \oplus K_*(\Bbbk)$. It can also be shown that the categories $\mathrm{perf}\text{–}\mathscr{R}_{\mathfrak{F}}(\chi)$ does not have exceptional objects, except in some special cases like $k_i = 0$.

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a1) two arrows $\{c_1, c_2\}$ from $\mathbf{1}$ to $\mathbf{2}$,

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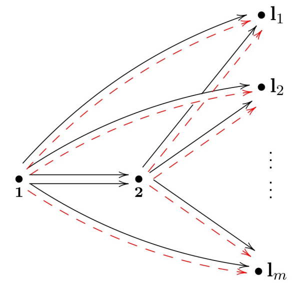

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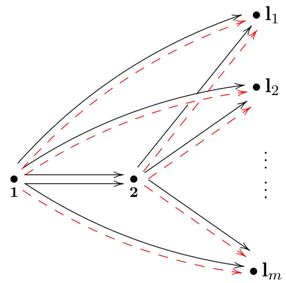

while the ideal of relations $J_{\mathfrak{F}}$ is generated by the elements:

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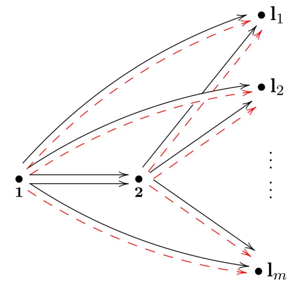

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Denote by $\mathscr{D}_{\mathfrak{F}}$ the quotient path algebra $\Bbbk\Phi_{\mathfrak{F}}/J_{\mathfrak{F}}$.

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$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{\ \mathrm{j}\ } & \mathrm{T} \\ \downarrow{\scriptstyle \mathrm{f}} & \overset{\mathrm{t}}{\swarrow} & \downarrow{\scriptstyle \mathrm{p}} \\ \mathrm{Y} & \xleftarrow{\ \mathrm{i}\ } & \mathrm{S} \end{array}$$

which is both cartesian and cocartesian.

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Prop: *There is a fully faithful functor* $\mathrm{perf}\text{-}\mathscr{R}_{\mathfrak{F}}(\chi_0) \hookrightarrow \mathrm{perf}\text{-}\mathscr{D}_{\mathfrak{F}}$, *where* $\chi_0 : \mathbb{Z}^{m+1} \to \mathbb{Z}$ *is such a map that* $\chi(\varepsilon_0) = 0$ *and* $\chi(\varepsilon_i) = 1$ *for any* i.

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Such smooth resolutions of rational curves can be generalized to the case of singular curves of arbitrary geometric genus g.

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Conclusion: Thus, in the case of an equidimensional family $\mathfrak{F}$ with $n = 2$ and $k = 1$, the DG category $\mathscr{P}erf-\mathscr{R}_{\mathfrak{F}}(\chi)$ and its triangulated category $\mathrm{perf}-\mathscr{R}_{\mathfrak{F}}(\chi)$ can be considered as a smooth proper noncommutative curve, because the category $\mathrm{perf}-\mathscr{R}_{\mathfrak{F}}(\chi)$ has the dimension equal to 1.

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THANK YOU!