Maurer-Cartan methods in perturbative quantum mechanics

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Motivation

Cohomological approach to QFT

$$\ldots \to \text{Gauge} \xrightarrow{Q_0} \text{Fields} \xrightarrow{Q_0} \text{EoM} \to \ldots$$

$$\text{Cohomology of } Q_0 \quad \leftrightarrow \quad \text{Physical states}$$

$$Q_0 \longrightarrow Q_0 + Q_{\text{int}}$$

QM scattering

Losev-S. `arXiv:2302.09464`

- Fixed energy E
- $Q = c(H_0 - E) + cV + \varepsilon S$
- Homotopy transfer $\longrightarrow$ symmetries of amplitudes

What if perturbations affect E? How to describe bound states?

Based on Losev-S. `arXiv:2401.17476`

Plan

1. Bring the Schrödinger equation to the Maurer-Cartan form
2. Describe the gauge group
3. Twist the differential
4. Find the cohomology of the twisted differential
5. Add a perturbation
6. Resolve obstructions
7. Represent corrections as diagrams

dgLie algebra of eigensystems

$$\psi + E + \text{odd parameters}$$

$$\downarrow$$

$$\hat{\mathscr{H}} = \begin{pmatrix} \mathbb{C}[\theta, c] \otimes \mathscr{H} \\ \mathbb{C}[c] \end{pmatrix} = \hat{\mathscr{H}}_0 \oplus \hat{\mathscr{H}}_1 \oplus \hat{\mathscr{H}}_2$$

- Differential $Q = \begin{pmatrix} cH & 0 \\ 0 & 0 \end{pmatrix}$
- Bracket $\{\Psi, \Phi\} = \begin{pmatrix} \Psi^{\mathrm{T}} \epsilon \Phi \\ 0 \end{pmatrix}$, where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Bracket on $\hat{\mathscr{H}}_1$

$$\{\Psi, \Phi\} = \{\begin{pmatrix} \theta\psi_1 + c\psi_2 \\ cE \end{pmatrix}, \begin{pmatrix} \theta\varphi_1 + c\varphi_2 \\ cW \end{pmatrix}\} = -\begin{pmatrix} c\theta(W\psi_1 + E\varphi_1) \\ 0 \end{pmatrix}$$

$(\hat{\mathscr{H}}, Q, \{\cdot, \cdot\})$ — dgLie algebra of eigensystems

Maurer-Cartan equation

$$\hat{\mathscr{H}}_1 \ni \quad \Psi = \begin{pmatrix} \theta\psi + c\varphi \\ cE \end{pmatrix} \text{ is a Maurer-Cartan element if it solves}$$

$$\hat{\mathscr{H}}_2 \ni \quad Q\Psi + \frac{1}{2}\{\Psi, \Psi\} = \begin{pmatrix} c\theta H\psi \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2c\theta E\psi \\ 0 \end{pmatrix} = \begin{pmatrix} c\theta(H\psi - E\psi) \\ 0 \end{pmatrix} = 0$$

Thus

- $\hat{\mathscr{H}}_1$ is the space of fields
- $\hat{\mathscr{H}}_2$ is the space of EoM
- $\hat{\mathscr{H}}_0$ must be the space of gauge fields

Gauge action

Gauge transformation: $\Psi \mapsto \Psi + Q\Phi + \{\Psi, \Phi\}$

Explicitly,

- energy is preserved: $E \mapsto E$
- wave function is scaled: $\psi \mapsto (1 + W)\psi$

Apparent symmetry of the linear equation became a gauge symmetry of the nonlinear one.

Twisting procedure

General result

Given $\Psi \in \mathrm{MC}(\hat{\mathscr{H}})$ we can define the twisted differential $\widetilde{Q} = Q + \{\Psi, \cdot\}$.

Explicitly: $\widetilde{Q} = \begin{pmatrix} c(H-E) & \theta\psi \\ 0 & 0 \end{pmatrix}$

Corollary

If $\Psi + \Phi \in \mathrm{MC}(\hat{\mathscr{H}})$, then $\widetilde{Q}\Phi + \frac{1}{2}\{\Phi, \Phi\} = 0$.

Cohomology

Useful notation: $H_E = H - E, \qquad \mathscr{K} = \ker H_E = \langle \psi \rangle \oplus \widetilde{\mathscr{K}}, \qquad \mathscr{I} = \operatorname{im} H_E$

$$H_{\widetilde{Q}} = \frac{\ker \widetilde{Q} = \begin{pmatrix} \mathscr{K} + \theta\mathscr{K} + c\mathscr{H} + c\theta\mathscr{H} \\ 0 \end{pmatrix}}{\operatorname{im} \widetilde{Q} = \begin{pmatrix} \theta\langle\psi\rangle + c\mathscr{I} + c\theta(\mathscr{I} + \langle\psi\rangle) \\ 0 \end{pmatrix}} = \begin{pmatrix} \mathscr{K} + \theta\widetilde{\mathscr{K}} + c\mathscr{K} + c\theta\widetilde{\mathscr{K}} \\ 0 \end{pmatrix}$$

NB

The important part of $H_{\widetilde{Q}}$ is $c\theta\widetilde{\mathscr{K}}$.

Homotopy

In QM scattering

Homotopy $h = G\partial_c$, where $G = \begin{cases} H_E^{-1}, & \mathscr{I} \\ 0, & \mathscr{K} \end{cases}$

It satisfied $h^2 = 0$ and $hQ + Qh = \mathbb{1} - \Pi_{H_Q}$.

Here we extend it to $h = \begin{pmatrix} G\partial_c & 0 \\ Y\partial_\theta & 0 \end{pmatrix}$, where $Y(\varphi) = (\psi, \varphi)$.

Once again we have $h^2 = 0$ and $h\widetilde{Q} + \widetilde{Q}h = \mathbb{1} - \Pi_{H_{\widetilde{Q}}}$.

Perturbation

So far we focused on the nonperturbed problem, so add $_0$ or $^{(0)}$ to all previous formulas.

New $Q = Q_0 + \lambda Q_1 = cH_0 + c\lambda V$.

$$(H_0 + \lambda V)\psi = E\psi \quad \Leftrightarrow \quad Q\Psi + \frac{1}{2}\{\Psi, \Psi\} = 0$$

As one does in perturbation theory, decompose

$$\Psi = \sum_{k=0}^{\infty} \lambda^k \begin{pmatrix} \psi^{(k)} \\ E^{(k)} \end{pmatrix} = \Psi^{(0)} + \hat{\Psi}$$

Expand and rearrange the Maurer-Cartan equation:

$$\widetilde{Q}_0 \hat{\Psi} = -\lambda Q_1 \Psi - \frac{1}{2}\{\hat{\Psi}, \hat{\Psi}\}$$

Is the r.h.s. $\widetilde{Q}_0$-exact?

Obstructions

First order in λ in the r.h.s.: $-\lambda c\theta V\psi^{(0)}$

But the degree 2 part of $H_{\widetilde{Q}_0}$ is $c\theta\widetilde{\mathscr{K}}$

Obstruction description

$V\psi^{(0)} \in \widetilde{\mathscr{K}}$ — precisely the condition from degenerate perturbation theory!

Corrections

Assume obstructions vanish. Then applying h_0 gives a recurrence relation:

$$\hat{\Psi} = -\lambda h_0 Q_1 \Psi^{(0)} - \lambda h_0 Q_1 \hat{\Psi} - \frac{1}{2} h_0 \{\hat{\Psi}, \hat{\Psi}\}$$

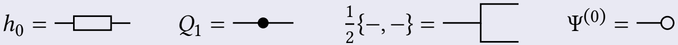
In order k:

$$\Psi^{(k)} = -h_0 Q_1 \Psi^{(k-1)} - \frac{1}{2} h_0 \sum_{m=1}^{k-1} \{\Psi^{(m)}, \Psi^{(k-m)}\}$$

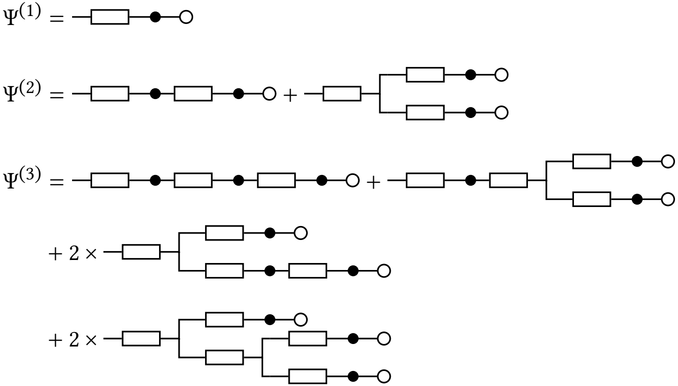
In particular:

$$\begin{aligned}
\Psi^{(1)} &= -h_0 Q_1 \Psi^{(0)} \\
\Psi^{(2)} &= -h_0 Q_1 \Psi^{(1)} - \frac{1}{2} h_0 \{\Psi^{(1)}, \Psi^{(1)}\} \\
\Psi^{(3)} &= -h_0 Q_1 \Psi^{(2)} - h_0 \{\Psi^{(1)}, \Psi^{(2)}\} \\
&\dots
\end{aligned}$$

Diagram rules



This allows us to rewrite the corrections in the form of tree diagrams:



Thank you for your attention!