Li-Yau sub-gradient estimates and Perelman-type entropy formulas for the heat equation in quaternionic contact geometry

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(joint work with Stefan Ivanov)

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Introduction – the Riemannian and CR cases

- Let (M^n, g) be a complete Riemannian manifold with Ricci curvature bounded from below by $-k$, where k is a non-negative constant, and u is a positive solution to the heat equation

$$\frac{\partial}{\partial t} u = \Delta u.$$

- In the prominent paper **P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156 (1986), 139-168**, Li and Yau obtained the following important gradient estimate for u:

$$||\nabla f||^2 - \alpha \frac{\partial}{\partial t} f \le \frac{n\alpha^2}{2t} + \frac{n\alpha^2 k}{2(\alpha - 1)}, \qquad (1)$$

where $\alpha > 1$ and $f = \ln u$.

- If $k = 0$, by letting $\alpha \to 1$, one have the estimate

$$||\nabla f||^2 - \frac{\partial}{\partial t} f \le \frac{n}{2t}. \qquad (2)$$

- The estimate (2) is sharp in the sense that the equality can be achieved by the fundamental solution of R^n. However, (1) is not sharp when $k > 0$. Finding sharp Li-Yau type gradient estimate for $k > 0$ is still an open problem. The Li–Yau gradient estimate (1) is an important tool in Geometric analysis. It provides the Harnack inequality for positive solution of the heat equation and Gaussian bounds for the heat kernel.

Introduction – the Riemannian and CR cases

- In his fundamental paper **Perelman G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159**, G. Perelman derived an entropy formula for Ricci flow. The formula turns out being of fundamental importance in the study of Ricci flow. The derivation of the entropy formula resembles the gradient estimate for the linear heat equation proved by Li and Yau on the linear parabolic equation.

- The corresponding problems in CR-geometry, namely the sub-parabolic Li-Yau CR gradient estimate and the Perelman CR entropy formula are developed in **Chang, S.-C., Kuo, T.-J. & Lai, S.-H., *Li-Yau gradient estimate and entropy formulae for the CR heat equation in a closed pseudohermitian* 3*-manifold*, J. Differential Geometry 89 (2011) 185-216**.

- The aim of this talk is to present the quaternionic contact (qc) counterpart of the Li-Yau gradient estimate and the Perelman's entropy.

Quaternionic contact structures – definition

Definition

A quaternionic contact structure on M^{4n+3} is the data of codimension three distribution H on M equiped with an $Sp(n)Sp(1)$–structure, i.e. we have

1. *a fixed conformal class $[g]$ of Riemannian metrics on H;*
2. *a 2-sphere bundle $\mathbb{Q}$ over M of almost complex structures, such that, we have $\mathbb{Q} = \{aI_1 + bI_2 + cI_3 : \ a^2 + b^2 + c^2 = 1\}$, where the almost complex structures $I_s : H \to H, \quad I_s^2 = -id_{|H}, \quad s = 1, 2, 3$, satisfy the commutation relations of the imaginary quaternions, $I_1 I_2 = -I_2 I_1 = I_3$, and which are hermitian compatible with the metric g, i.e. $g(I_s\cdot, I_s\cdot) = g(\cdot, \cdot), \quad s = 1, 2, 3$;*
3. *H is the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and the following compatibility condition holds*

$$2g(I_s X, Y) = d\eta_s(X, Y) := 2\omega_s(X, Y), \quad s = 1, 2, 3, \quad X, Y \in H.$$

- Given η (and H) there exists at most one triple of a. c. str. and metric g that are compatible.
- Rotating η we obtain the same qc-structure.
- Given the horizontal space H and a metric g on it, there exists at most one sphere bundle of associated contact forms with a corresponding sphere bundle $\mathbb{Q}$ of a. c. str.

Quaternionic contact structures – definition

A quaternionic contact (qc) structure, introduced by O. Biquard:

- appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space;
- appears naturally in connection with the quaternionic contact Yamabe problem (a particular case of this problem amounts to finding the extremals and the best constant in the L^2 Folland-Stein Sobolev-type embedding on the quaternionic Heisenberg group);
- is a special kind of sub-Riemannian geometry;
- is a generalization of 3-Sasaki manifold;

Examples:

- 3-Sasakian manifolds M: The metric $g_{con} = t^2 g + dt^2$ of the cone $C = M \times \mathbb{R}^+$ is a hyperkähler, i.e. $Hol(\nabla^{g_{con}}) \subset Sp(n+1)$;
- The quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$;

Theorem (O. Biquard, 2000)

For a qc manifold $(M, g, \mathbb{Q}, \eta)$ of dimension bigger than seven ($n > 1$), there exists a unique supplementary distribution V of H in TM and a linear connection ∇ on M, s.t.,

1. *∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$–structure on H, i.e. $\nabla g = 0, \nabla\sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X,Y) = -[X,Y]_{|V}, \quad X, Y \in H$;*
2. *for $R \in V$, the endomorphism $T(R,.)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^{\perp} \subset gl(4n)$;*
3. *the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.*

Note: V is generated by the Reeb vector fields (ξ_1, ξ_2, ξ_3) determined by

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \qquad (\xi_s \lrcorner d\eta_t)_{|H} = -(\xi_t \lrcorner d\eta_s)_{|H}.$$

If the dimension of M is seven, $n = 1$, the above conditions do not always hold. Duchemin shows that if we assume, in addition, the existence of Reeb vector fields as above, then the Biquard result holds.
Using the triple of Reeb vector fields we extend the metric g on H to a Riemannian metric on TM by requiring $span\{R_1, R_2, R_3\} = V \perp H$, i. e. $g \oplus \sum(\eta^i)^2$ is a Riemannian metric on the entire TM.

The torsion and the curvature tensors

- torsion: $T(A,B) = \nabla_A B - \nabla_B A - [A,B], \quad T(A,B,C) = g(T(A,B),C)$;
- torsion endomorphism: $T_R = T(R,\cdot) : H \to H, \quad R \in V$;
- decomposition of the torsion endomorphism: $T_R = T_R^0 + b_R$ into its symmetric part T_R^0 and skew-symmetric part b_R;
- symmetric, trace-free endomorphism on $H : \tau = (T^0_{R_1} I_1 + T^0_{R_2} I_2 + T^0_{R_3} I_3)$;
- representation of the skew-symmetric part: $b_{R_i} = I_i \mu$, where μ is a traceless symmetric (1,1)-tensor on H which commutes with I_1, I_2, I_3;
- curvature: $R(A,B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C, \quad R(A,B,C,D) = g(R(A,B)C,D)$;
- qc-Ricci tensor: $Ric(X,Y) = tr_H\{Z \mapsto R(Z,X)Y\} = R(e_a,X,Y,e_a)$ for $e_a, X, Y \in H$;
- qc-Ricci type tensors: $\rho_s(A,B) = \frac{1}{4n} R(A,B,e_a,I_s e_a); \quad \zeta_s(A,B) = \frac{1}{4n} R(e_a,A,B,I_s e_a)$;
  $\sigma_s(A,B) = \frac{1}{4n} R(e_a, I_s e_a, A, B)$;
- qc-scalar curvature: $S = tr_H\, Ric = Ric(e_a,e_a)$;
- The qc-Einstein condition: $Ric(X,Y) = fg(X,Y)$;

Some basic notions and notations

- Denote by $(e_1, e_2, \dots, e_{4n})$ an orthonormal frame of the horizontal space H and $\varphi \in \mathcal{F}(M)$;
- The sub-Laplacian: $\Delta_b \varphi := -\nabla^2 \varphi(e_a, e_a)$;
- The horizontal gradient: $\nabla_b \varphi := \sum_{a=1}^{4n} d\varphi(e_a) e_a$;
- The vertical gradient: $\nabla_v \varphi := \sum_{s=1}^{3} d\varphi(\xi_s) \xi_s$;
- The qc heat equation (compact M):

$$\frac{\partial}{\partial t} u = -\Delta_b u, \tag{3}$$

where $u \in \mathcal{F}(M \times [0, \infty))$;

Main results – part I

Our first main result states as follows:

Theorem (Ivanov - P., 2024)

Let $(M, g, \mathbb{Q})$ be a compact qc manifold of dimension $4n+3$ and the positivity condition

$$2(n+2)Sg(X,X) + 2nT^0(X,X) + 4(n+4)U(X,X) \geq 0 \tag{4}$$

holds. Suppose that $u(x,t)$ is a positive solution of (3), satisfying

$$(\nabla^3 u)(e_a, e_a, \nabla_v u) = -\nabla_v u(\Delta_b u). \tag{5}$$

For $f(x,t) = \ln u(x,t)$ the following sub-gradient estimate holds

$$|\nabla_b f|^2 - \alpha \frac{\partial}{\partial t} f + \frac{8n}{3} t |\nabla_v f|^2 \leq \frac{2n\alpha^2}{t}, \quad \alpha = \frac{9+2n}{2n}. \tag{6}$$

As a consequence of the upper theorem we get the next

Corrolary

Let $(M, g, \mathbb{Q})$ be a compact qc-Einstein manifold of dimension $4n+3$ with non-negative constant qc scalar curvature, $S \geq 0$. Supose that $u(x,t)$ is a positive solution of (3). Then $f(x,t) = \ln u(x,t)$ satisfies the sub-gradient estimate (6).
In particular, on a compact 3-Sasakian manifold the function $f(x,t)$ satisfies the sub-gradient estimate (6).

Main results – part I

Our second main result concerns the case when the a priori lower bound (4) of the upper theorem is replaced by a negative constant, namely, we assert the following

Theorem (Ivanov - P., 2024)

Let $(M, g, \mathbb{Q})$ *be a compact qc manifold of dimension* $4n+3$ *satisfying the positivity condition*

$$2(n+2)Sg(X,X) + 2nT^0(X,X) + 4(n+4)U(X,X) \geq -kg(X,X), \tag{7}$$

where $k > 0$ *is a constant. Suppose that* $u(x,t)$ *is a positive solution of* (3)*, satisfying* (5)*. Then the following sub-gradient estimate for* $f(x,t) = \ln u(x,t)$ *holds:*

$$|\nabla_b f|^2 - \alpha_k \frac{\partial}{\partial t} f < \frac{n\alpha_k^2(k+2)}{t} + \frac{8n^2\alpha_k^2(k+2)}{k+3}, \quad \alpha_k = \frac{9+2n(k+1)}{2n}. \tag{8}$$

As a simple consequence of the upper theorem we obtain the following

Corrolary

Let $(M, g, \mathbb{Q})$ *be a compact qc-Einstein manifold of dimension* $4n+3$ *with negative qc scalar curvature* $S = -\frac{k}{2(n+2)}$*, where* $k > 0$ *is a constant. Suppose that* $u(x,t)$ *is a positive solution of* (3)*. Then* $f(x,t) = \ln u(x,t)$ *satisfies the sub-gradient estimate* (8)*.*

Nash-type and Perelman-type functionals

Furthermore, following the Riemannian and CR cases, we define the *Nash–type functionals*

- $\mathcal{N}(u,t) = -\int_M (\ln u) u \, Vol_\eta$,
- $\tilde{\mathcal{N}}(u,t) = \mathcal{N}(u,t) - 2n\alpha^2 a[\ln(4\pi t) + 1]$,

as well as the *Perelman–type functionals*

- $\mathcal{W}(u,t) = \int_M [t|\nabla_b \varphi|^2 + \varphi - 4n\alpha^2 a] u \, Vol_\eta$,
- $\tilde{\mathcal{W}}(u,t) = \mathcal{W}(u,t) + 4nt^2 \int_M |\nabla_v \varphi|^2 u \, Vol_\eta$.

In the upper expressions, $u(x,t)$ is a positive solution of the qc heat equation (3), normalized as

$$\int_M u \, Vol_\eta = 1, \qquad \text{expressed by} \tag{9}$$

$$u(x,t) = \frac{e^{-\varphi(x,t)}}{(4\pi t)^{2n\alpha^2 a}} \tag{10}$$

for a suitable function $\varphi(x,t) : M \times [0,\infty) \to \mathbb{R}$, where $\alpha = \frac{9+2n}{2n}$ and a is a real constant to be determined.

Main results – part II

The second cycle of results is related to two Perelman-type entropy formulas concerning $\tilde{\mathbb{N}}(u,t)$ and $\tilde{\mathbb{W}}(u,t)$. The first one is the following

Theorem (Ivanov - P., 2024)

Let $(M, g, \mathbb{Q})$ be a compact qc-Einstein $(4n+3)$-dimensional manifold with non-negative qc scalar curvature. Let $u(x,t)$ be a positive solution of (3), satisfying (9). Then we have

$$\frac{d}{dt}\tilde{\mathbb{N}}(u,t) = \int_M \Big[|\nabla_b \varphi|^2 + \alpha \frac{\partial}{\partial t}\varphi + \frac{2n(\alpha-1)\alpha^2 a}{t}\Big] u \, Vol_\eta \le 0 \tag{11}$$

for $t \in (0,\infty)$ and $a \ge 1$.
In particular, on a compact 3-Sasakian manifold any positive solution to the qc heat equation (3) normalized by (9) satisfies the inequality (11).

Main results – part II

As a consequence of the last theorem we obtain the following integral version of the sub-gradient estimate:

Corrolary

Let $(M, g, \mathbb{Q})$ *be a compact qc-Einstein manifold of dimension* $4n+3$ *with non-negative qc scalar curvature. Suppose that* $u(x,t)$ *is a positive solution of* (3)*. Then there exists a positive constant* C *s.t. the following estimate holds:*

$$\int_M |\nabla_b u^{\frac{1}{2}}|^2 \, Vol_\eta \leq \frac{C}{t}. \tag{12}$$

In particular, on a compact 3-Sasakian manifold any positive solution $u(x,t)$ *of the qc heat equation* (3) *satisfies the integral estimate* (12)*.*

Main resuts – part II

The second Perelman–type entropy formula is the content of the next

Theorem (Ivanov - P., 2024)

Let $(M, g, \mathbb{Q})$ be a compact qc-Einstein manifold of dimension $4n+3$ with non-negative qc scalar curvature. Let $u(x,t)$ be a positive solution of (3), satisfying (9). Then we have

$$\frac{d}{dt}\tilde{\mathcal{W}}(u,t) \le -2t\int_M u|(\nabla^2\varphi)_{[-1][sym]}|^2\, Vol_\eta \\ -\frac{t}{2n}\int_M u(\Delta_b\varphi)^2\, Vol_\eta - 4(n+2)St\int_M u|\nabla_b\varphi|^2\, Vol_\eta + \frac{2(2n-na+6)\alpha^2}{t} \le 0 \qquad (13)$$

for $t \in (0,\infty)$ and $a \ge \frac{2n+6}{n}$.
In particular, on a compact 3-Sasakian manifold any positive solution to the qc heat equation (3) normalized by (9) satisfies the inequality (13).

THANK YOU FOR THE ATTENTION!