A monoidal Bondal-Orlov reconstruction theorem

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1. Based on work done during my PhD under the supervision of Carlos Simpson.

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2. Current joint work with Artan Sheshmani.

Objects of interest

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I want to understand X by looking at the bounded derived category $D^b(X)$.

Chronology of the problem I

1. 1962 - Gabriel shows that $Coh(Y)$ is a complete invariant for noetherian schemes Y.

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2. 1981 - Mukai discovers the existence of non-trivial Fourier-Mukai partners, abelian varieties X, X' such that $D^b(Coh(X)) \simeq D^b(Coh(X'))$, yet $X \ncong X'$.
3. 2001 - Bondal and Orlov show this does not happen if X is such that it has ample canonical or anti-canonical bundle.

Chronology of the problem II

1. 2002-2005 - Balmer develops a construction which inputs a tensor triangulated category $(T, \otimes, \mathbb{U})$ and outputs a locally ringed space $Spec(T, \otimes, \mathbb{U})$. He shows that $Spec(Perf(X), \otimes^{\mathbb{L}}_X, \mathscr{O}_X) \cong X$ for topologically noetherian schemes X.

The natural questions

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No, and no.
Liu and Sierra showed that if $Perf(X) \simeq D^b(RepQ)$, the derived tensor product of quiver representations $\otimes_Q^{\mathbb{L}}$ gives $|Spec(Perf(X), \otimes_Q^{\mathbb{L}})| \in \mathbb{N}$.

Example

If $X = \mathbb{P}^1$ then by Bondal-Orlov, there are no non-trivial Fourier-Mukai partners of X. But $Perf(\mathbb{P}^1) \simeq D^b(RepK_1)$, where K_1 is the Kronecker quiver with two vertices and two edges.

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Serre functors

Definition

Let T be a triangulated category. A Serre functor $S : T \to T$ is an additive functor such that

$$Hom_T(A, B) \xrightarrow{\cong} Hom_T(B, S(A))^*$$

Example

If X is a smooth proj. variety of dimension n then $\_ \otimes_X^{\mathbb{L}} \omega_X[n]$ is a Serre functor.

Almost ample sequences

Definition

Let T be a triangulated category and $(\otimes, \mathbb{U})$ a TTC on T. We say that a collection of objects $\Omega \subseteq T$ is an spanning class if the following holds:

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- If $X \in T$ is such that $Hom_T(\pi(B), X[j]) = 0$ for all $B \in \Omega$ and $j \in \mathbb{Z}$ then $X \cong 0$.
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General type varieties

Combining everything

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- $\{\omega_X^{\otimes i}\}$ is an spanning class.
- The collection $\{\omega_X^{\otimes i}\}$ "generate" $D^b(X)$. Every complex $A^\bullet$ has a resolution with terms

$$\bigoplus_j \omega_{X^*}^{\otimes^{\mathbb{L}}_{X^*} i}$$

The theorem

Theorem ([Tol24])

Let X be a smooth proj. variety of dimension n with ample canonical bundle ω_X. If ω_X is an invertible object for a TTC $(\boxtimes, \mathscr{O}_X)$ on $D^b(X)$ then $\boxtimes$ and $\otimes_X^{\mathbb{L}}$ coincide on objects.

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Corollary ([Tol24])

Let X be a variety with ample canonical bundle and let $(\boxtimes, \mathscr{O}_X)$ be a TTC on $D^b(X)$. If $Spec(\boxtimes)$ is a smooth projective variety and we have an equivalence $D^b(X) \simeq D^b(Spec(\boxtimes))$ then $X \cong Spc(\boxtimes)$.

The setting

Can we give a relative version of the monoidal Bondal-Orlov reconstruction theorem? Calabrese gives in [Cal18] a proof for relative, singular spaces while considering twisted derived categories.

How to say this for the monoidal case?

We want to consider $\pi : X \to S$ a smooth projective variety, faithfully flat over a quasi-compact quasi-separated scheme S, whose fibers along any point $s \in S$ have ample (anti-)canonical bundle, and put a TTC $(\boxtimes, \mathbb{U})$ on $D^b(X)$ which

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1. Captures the data of the structure map $\pi : X \to S$.
2. Restricts to the derived category of the fibers $D^b(X_s)$.

Solutions

1. We need to impose some linearity condition wrt $\otimes_S^{\mathbb{L}}$: There is a projection formula, $\pi_*(\pi^* E^\bullet \boxtimes P^\bullet) \cong E^\bullet \otimes_S^{\mathbb{L}} \pi_* P^\bullet$ for any $E^\bullet \in D^b(S)$ and every $P^\bullet \in D^b(X)$.

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2. But, how to restrict to fibers? Is there an induced TTC structure on $D^b(X_U = X \times_S U)$ or $D^b(X_s)$? If so, how are these related?

Transporting TTCs

A general construction we can perform is, if $\boxtimes$ is a TTC on a triangulated category T', and $F : T \to T'$ has a right adjoint $G : T' \to T$ then we can write

$$(\_) \boxtimes_{FG} (\_) := G(F(\_) \boxtimes F(\_)) : T \times T \to T$$

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Immediate problem

This transported structure $\boxtimes_{FG}$ does not need to be a TTC on T.

Magmoidal structures

Silver lining

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Definition

We say $\boxtimes$ is reasonable with respect to $F \dashv G$ if

1. $F(\mathbb{U})$ acts as a unit for $\boxtimes_{FG}$.
2. $\boxtimes_{FG}$-invertible objects have strong duals.
3. If X is $\boxtimes_{FG}$-invertible then $X \boxtimes_{FG}$ is full.

The dg-world

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A well known problem with triangulated categories is that they do not behave that well in families. So let us move to the higher-categorical language of dg-categories. Recall that the triangulated category $D^b(X)$ has a dg-enhancement, which means there is a dg-category $D^b_{dg}(X)$ such that $H^0(D^b_{dg}(X)) \simeq D^b(X)$ as triangulated categories.

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But what about the tensor triangulated category structures? Do they have enhancements?

Theorem (Eilenberg-Watts theorem)

Let A, B be rings, there is an equivalence between the category $Fun_{add,cc}(R-Mod, S-Mod)$ of additive cocontinous functors between the categories of R-Modules and S-Modules, and the category $(R, S)-Bimod$ of $R-S$-bimodules (or $R^{op} \otimes S$-modules).

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We want a similar theorem for dg-categories enhancing derived categories of sheaves.

DK model structure

For this to work, we need to use some homotopy theory. Tabuada [Tab05] introduced a model category structure on the category of DG-categories.

Rough definition

We will declare two DG-categories $\mathscr{T}, \mathscr{T}'$ to be weakly equivalent if there is $\mathscr{F} : \mathscr{T} \to \mathscr{T}'$ inducing quasi-isomorphisms $Hom_{\mathscr{T}}(x, y) \to Hom_{\mathscr{T}}(F(x), F(y))$, and $H^0(\mathscr{F})$ is essentially surjective.

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There is a homotopy category H_{qe} resulting from this identification.

Morita theorem for DG-categories

In [Toë07], Toën constructed a derived tensor product $\otimes^{\mathbb{L}}$ of DG-categories on H_{qe}, $\otimes^{\mathbb{L}}_X$ and showed that the Morita theorem for DG-categories holds true when one passes to the homotopy category.

Theorem ([Toë07])

Let $\mathscr{T} = D^b_{dg}(X)$ and $\mathscr{T}' = D^b_{dg}(X')$ be two dg-enhancements of $D^b(X)$ and $D^b(X')$ for varieties X, X', then there exists a natural isomorphism in H_{qe}

$$\mathbb{R}\underline{Hom}_c(\widehat{\mathscr{T}}, \widehat{\mathscr{T}'}) \simeq \widehat{\mathscr{T}^{op} \otimes^{\mathbb{L}} \mathscr{T}'}$$

n-fold DG-bimodules

Definition

Let $\mathscr{T}$ be a DG-category. An n-fold DG-bimodule over $\mathscr{T}$ is a DG-module $\mathscr{F} \in \mathscr{T}^{\otimes n} \otimes \mathscr{T}^{op} - Mod$.

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In particular a 0-fold DG-bimodule is nothing but a $\mathscr{T}^{op} -$ module and a 1-fold bimodule is what we usually call a bimodule over $\mathscr{T}$. We consider the dg-category $Bimod^n_{dg}(\mathscr{T})$ of n-fold dg-bimodules over $\mathscr{T}$.

Lifting TTC's

Theorem

Suppose $\mathscr{T}$ is equivalent in H_{qe} to A_{pe}, the category of perfect DG-modules over a DG-algebra A. Let $\boxtimes : H^0(\mathscr{T}) \times H^0(\mathscr{T}) \to H^0(\mathscr{T})$ be an exact functor in each variable. Suppose that for every object $M \in H^0(\mathscr{T})$, the triangulated functors

$$M \boxtimes \_ : H^0(\mathscr{T}) \to H^0(\mathscr{T})$$
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both have unique DG-enhancements R_M and L_M respectively. Then $L_A(A)$ is a 2-fold DG-bimodule and for any $N \in \mathscr{T}$ we have

$$H^0(L_A(A) \otimes M \otimes N) \simeq M \boxtimes N.$$

What does this mean?

Remark

The previous result allows us to encode a tensor product $\boxtimes : T \times T \to T$ in a TTC as a 2-fold DG-bimodule over a DG-enhancement $\mathscr{T}$ of T.

In applications in Algebraic Geometry, both $\mathscr{T}$ and $\boxtimes$ would usually satisfy the theorem's hypothesis.

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In applications in Algebraic Geometry, both $\mathscr{T}$ and $\boxtimes$ would usually satisfy the theorem's hypothesis.

The proof of the theorem uses strongly Toën's Morita theorem for DG-categories.

Pseudo DG-tensor structures

Let us present the following definition in obvious analogy with the usual (lax) symmetric monoidal category axioms:

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2. An object $U \in \mathscr{T}^{op} - Mod$ called the unit.

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1. A 2-fold DG-bimodule $\Gamma \in Bimod_{dg}^{2}(\mathscr{T})$
2. An object $U \in \mathscr{T}^{op} - Mod$ called the unit.
3. Morphisms of DG-bimodules
   $\alpha_{X,Y,Z} : \Gamma_{X,\Gamma} \otimes \Gamma_{Y,Z} \to \Gamma_{\Gamma,Z} \otimes \Gamma_{X,Y} \in Bimod_{dg}^{3}(\mathscr{T})$.

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4. A morphism of DG-bimodules $\ell_X : \Gamma_{U,X} \otimes U \to \mathscr{T} - Mod \in Bimod^1_{dg}(\mathscr{T})$.
5. A morphism of DG-bimodules $r_X : \Gamma_{X,U} \otimes U \to \mathscr{T} - Mod \in Bimod^1_{dg}(\mathscr{T})$.

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5. A morphism of DG-bimodules
   $r_X : \Gamma_{X,U} \otimes U \to \mathscr{T} - Mod \in Bimod^1_{dg}(\mathscr{T})$.
6. A morphism $c_{X,Y} : \Gamma_{X,Y} \to \Gamma_{Y,X}$ of DG-bimodules.

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6. A morphism $c_{X,Y} : \Gamma_{X,Y} \to \Gamma_{Y,X}$ of DG-bimodules.

We require that the morphisms u_X and $c_{X,Y}$ are all isomorphisms when passing to the homotopy category $H^0(\mathscr{C}(k))$, for all $X, Y \in \mathscr{T}^{op} - Mod$.

Continuation

Definition

Plus..

1. (Unit) A morphism $\mu \in Hom^{-1}(\Gamma_{X,\Gamma} \otimes \Gamma_{U,Y}, \Gamma_{\Gamma,Y} \otimes \Gamma_{X,U})$ such that $\ell^0_X \otimes Id_Y \circ \alpha^0_{X,U,Y} - Id_X \otimes \ell^0_Y = d(\mu)$
2. (Symmetry) The composition $c_{X,Y} \circ c_{Y,X}$ is the identity in $H^0(\mathscr{T} - Mod)$.
3. (Unit symmetry) There is $\kappa \in Hom^{-1}(\Gamma_{X,U}, X)$ such that $\ell_X \circ c_{X,U} - r_X = d(\kappa)$.
4. Etc..

Lifting tensor triangulated structures

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Proposition [TC23]

A perfect pseudo DG-magmoidal structure Γ on a DG-category $\mathscr{T}$ induces a magmoidal triangulated category structure on $H^0(\mathscr{T}_{pe})$.

Now we are talking!

By translating the TTC structure in terms of modules of a certain kind at the level of dg-categories, we can more easily perform useful common constructions which are purely algebraic, like taking (co)limits!

A stack of dg-categories

In [HS98] Hirschowitz and Simpson, (and then Toën-Vaquié ([TV07]) construct a (derived) stack $\mathbb{A}_{pe}$ of simplicial categories of perfect complexes. One can adjust this definition and see that the assignment on affine opens

$$U \subseteq S \mapsto \mathbb{A}_{pe}(U) := Perf^{dg}(X_U)$$

glues into a higher stack with values in the ∞-category of dg-categories.

Stalks around a point

By using the stalk $\mathbb{A}_{pe}$, we can define "the stalk" around a point $s \in S$ of this assignment. We write simply

$$(\mathbb{A}_{pe})_s = Colim_{U \ni s} \mathbb{A}_{pe}(U)$$

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Geometric reasonableness

We need a compatibility condition. A MTC structure $\boxtimes$ or a dg-enhancement Γ of it is geometrically reasonable if

1. It is reasonable wrt $I_U^* \dashv I_U *$ for $I_U : X_U \hookrightarrow X$.

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1. It is reasonable wrt $I_U^* \dashv I_U *$ for $I_U : X_U \hookrightarrow X$.
2. The restriction functors $\mathbb{A}_{pe}(V) \to \mathbb{A}_{pe}(U)$ are "monoidal".

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2. The restriction functors $\mathbb{A}_{pe}(V) \to \mathbb{A}_{pe}(U)$ are "monoidal".

$$\phi_{UV}(\_ \boxtimes_V \_) \simeq \_ \boxtimes_U \_$$

One more stack

Using the same formalism of Toën-Vaquié, we can also construct the Internal Hom stack $\mathbb{R}\underline{Hom}(\mathbb{A}_{pe}^{\otimes 2}, \mathbb{A}_{pe})$ which parametrizes 2-fold dg-bimodules over the base scheme S.
On each $U \subseteq S$ we obtain the simplicial set of maps between the stacks $\mathbb{A}_{pe}^{\otimes 2}$ and $\mathbb{A}_{pe}$.

Time to assemble

We start from the data of a S-linear tensor structure $(\boxtimes, \mathbb{U})$ on a smooth proj. $\pi : X \to S$ faithfully flat over S so that X_s has ample canonical or anti-canonical bundles for every $s \in S$.

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Theorem ([ST24])

If $\boxtimes$ is reasonable with respect to $I_s^* \dashv I_{s\,*}$, $I_s^*\mathbb{U} \simeq \mathscr{O}_{X_s}$ and ω_{X_s} is $\boxtimes_s$-invertible, then $\boxtimes_s$ coincides on objects with $\otimes^{\mathbb{L}}_{X_s}$.

Intermediate notation/definition

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1. $\mathfrak{m}_U = \{P^\bullet \mid SupphP^\bullet = \emptyset\}$

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2. $\mathfrak{m}_s = Colim_{U \ni s}\mathfrak{m}_U$
3. $\mathbb{A}_{pe}(s) := Perf^{dg}(X_s)$

Time to assemble II

From fibers to stalks

There are diagrams in the ∞-category of dg-categories:

$$
\begin{array}{ccccc}
 & \swarrow & \mathbb{A}_{pe}(U) & \searrow & \\
 & & \downarrow & & \\
(\mathbb{A}_{pe})_s & \longrightarrow & \mathbb{A}_{pe}(s) & \longleftarrow & \mathbb{A}_{pe}(U)/\mathfrak{m}_U \\
 & \searrow & \uparrow & \swarrow & \\
 & & (\mathbb{A}_{pe})_s/\mathfrak{m}_s & &
\end{array}
$$

Nakayama

We use [Zim23] to argue using a dg-version of the Nakayama lemma for fg modules over a dg-algebra A and conclude that the vanishing of an object $P \in (\mathbb{A}_{pe})_s$ is equivalent to the vanishing when passing to the quotient/the fiber $\mathbb{A}_{pe}(s)$.

Another diagram

From fibers to stalks again

$$\begin{array}{ccc} & & \mathbb{R}\underline{Hom}(\mathbb{A}_{pe}(s)^{\otimes 2}, \mathbb{A}_{pe}(U)) \\ & \swarrow & \downarrow \\ \mathbb{R}\underline{Hom}(\mathbb{A}_{pe}(s)^{\otimes 2}, (\mathbb{A}_{pe})_s) & \longrightarrow & \mathbb{R}\underline{Hom}(\mathbb{A}_{pe}(s)^{\otimes 2}, \mathbb{A}_{pe}(s))_{ac} \\ & \searrow & \uparrow \\ & & \mathbb{R}\underline{Hom}(\mathbb{A}_{pe}(s)^{\otimes 2}, (\mathbb{A}_{pe})_s)/H\mathfrak{m}_s \end{array}$$

The actual theorem

Theorem

Let $X \to S$ smooth proj faithfully flat over qsqc S with a point $s \in S$ s.t. X_s has ample canonical or anti-canonical bundle. Let $(\boxtimes, \mathbb{U})$ be an S-linear TTC over $Perf(X)$, which is geometrically reasonable and reasonable with respect to the pair $l_s^* \dashv l_{s*}$. If ω_{X_s} is $\boxtimes_s$-invertible and $l_s^*(\mathbb{U}) \simeq \mathscr{O}_{X_s}$ then there is an open affine subset $U \subseteq S$ such that $\boxtimes_U$ coincides on objects with $\otimes^{\mathbb{L}}_{X_U}$.

Relative Monoidal Bondal-Orlov

Theorem

Suppose $X \to S$ is a smooth proj. variety faithfully flat over a qsqc base scheme S. Suppose for every $s \in S$ X_s has an ample canonical or anti-canonical bundle. Let $(\boxtimes, \mathbb{U})$ be as above. Suppose that $Spec(\boxtimes)$ is a smooth projective variety, faithfully flat over S such that $D^b(Spec(\boxtimes)) \simeq D^b(X)$, then $X \cong Spec(\boxtimes)$ as S-schemes.

Thank you!

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