Degenerations, derived Lagrangians and Categorification of DT invts

(.... and S-duality conjecture!)

Joint Work with Baranovsky, katzarkov, kontsevich

Borisov, katzarkov, Yau

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Motivation (S-duality modularity) Conjecture

Let X : CY3, $H^1(\mathcal{O}_X)=0$

Assume $\mathrm{Pic}(X)$ is generated by an ample divisor L

fix $k \in \mathbb{Z}_{>0}$, let $H \in |kL|$

Smooth element.

l: generator of $H^4(X;\mathbb{Z}) \cong \mathbb{Z}$

Fixed Chern character   fix $i, n \in \mathbb{Z}$

$$\bar{c} = \bar{c}(i,n) = \left(0, H, \frac{H^2}{2} - i\ell, \chi(\mathcal{O}_H) - H \cdot \frac{td}{2}(X) - n\right)$$

Let $M_{\bar{c}}(X)$: Moduli Space of Gieseker S.S Sheaves

$F \in Coh(X)$, $ch(F) = \bar{c}$ (i

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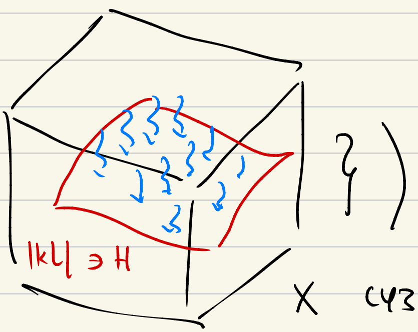
$M(\{$  $\})$

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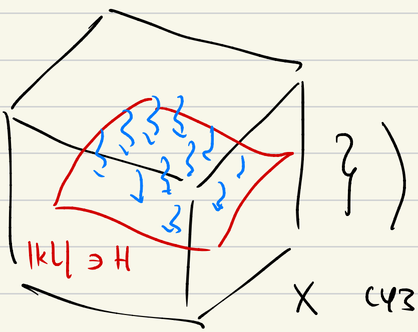
$M(\{$  $\})$ $ch(F) = \bar{c}(i,n)$

let $\overline{DT}(\bar{c}(i,n)) \in \mathbb{Q}$: Joyce-Song generalized DT invt.

Partition function:

$$Z_i^H(q) = \sum_{n \in \mathbb{Z}} \overline{DT}(\bar{c}(i,n))\, q^n$$

Rmk Tensoring by $\mathcal{O}(\pm L)$ induces isomorphisms on $\mathcal{M}_{\tau}(X)$

hence $Z_i^H(q)$ and $Z^H_{i\pm kL^3}(q)$ only differ by a

shift in power of q.

<u>Rmk</u> Tensoring by $\mathcal{O}(\pm L)$ induces isomorphisms on $\mathcal{M}_{\tau}(X)$

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shift in power of q.

$\mathcal{S}$-duality Conjecture

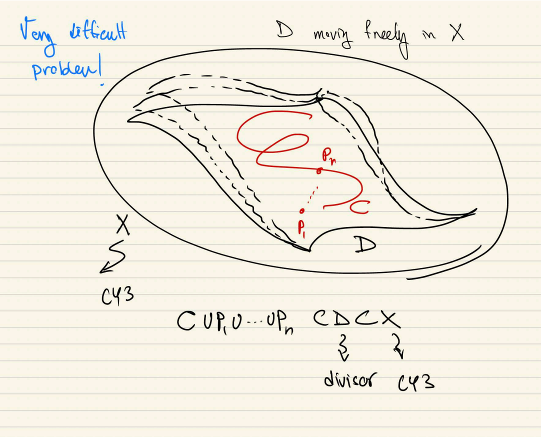
(Gaiotto-Strominger-Yin)

$$\left(q^{a_i} Z_i^H(q)\right)_{i\geq 0}^{kL^3-1} = \left(\underbrace{q^{a_0} Z_0^H(q)}, \underbrace{q^{a_1} Z_1^H(q)}, \cdots, \underbrace{q^{a_{kL^3-1}} Z^H_{kL^3-1}(q)}\right)$$

is a holomorphic vector valued modular form of

weight $\frac{-3}{2}$

Where $a_i = \dfrac{(2i+H)^3}{8H^3} - \dfrac{H^3}{8} - \dfrac{\chi(H)}{24} \in \mathbb{Q}$.



Counting curves on surfaces in Calabi-Yau threefolds, (with Amin Gholampour and Richard P. Thomas), Mathematische Annalen, Volume 360, Issue 1-2, pp 67-78 (2014), arXiv:1309.0051.

Obtain modular partition function? Almost!

$$Z = \sum_{\beta, n} = \blacksquare + \blacksquare + \square + \square + \square + \cdots \blacksquare + \square + \blacksquare + \cdots$$

To prove modularity we need:

1. Generalize from rank 1, ideal sheaf counting to rank one general sheaf counting
2. Compute higher rank sheaf counting from rank 1 sheaf ocunting (Feyzbakhsh, Thomas)

Rmk When sheaf F = ideal sheaf 1-dim'l subscheme

$Z \subset H$; in joint work with Gholampour - Thomas (2014)

we defined invts $N^{H}_{\beta,n}$ which govern contribution

of $\mathrm{Hilb}_{\beta,n}\left(\tilde{H}/|H|\right)$ to $DT_{\bar{C}}(X)$.

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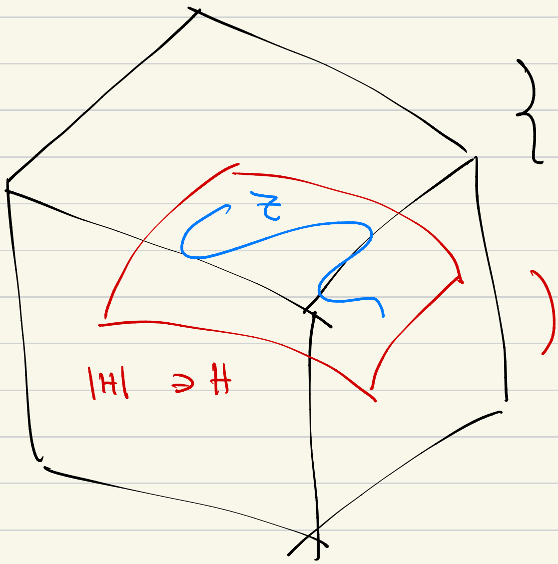
universal Hypersurface.

$\mathrm{Hilb}_{\beta,n}\left(\tilde{H}/|H|\right) = M\Big($  $\Big)$

Needed a positivity condition for def-obs thy to work

$$H^i(X, \mathcal{I}_{Z/X} \otimes \mathcal{O}_X(H)) = 0 \quad \forall i > 0$$

$$N^H_{\beta,n} = \int_{[M]^{vir}} 1 \longleftrightarrow \deg [M]^{vir} = 0$$

To prove S-duality Conjecture

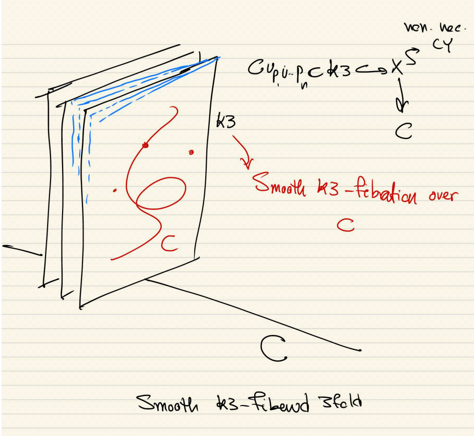
Step 0: Contribution of $N^{H}_{\beta,n}$ to $DT(\mathcal{X})$ (GST 2014)

Step 1: Contribution of all Sheaves of $rk=1$ with Support
and investigating modularity.
Still open

Step 2: Contribution of Higher rank Sheaves with
Support on H and investigating modularity → Feyzbaksh & Thomas (2020)
Still open

Wall crossing ← (2020)

rank 0 → rank 1 → rank > 1



Donaldson-Thomas Invariants of 2-Dimensional sheaves inside threefolds and modular forms, (with Amin Gholampour), Advances in Mathematics, Vol. 326, No. 21, p. 79-107 arXiv:1309.0050.

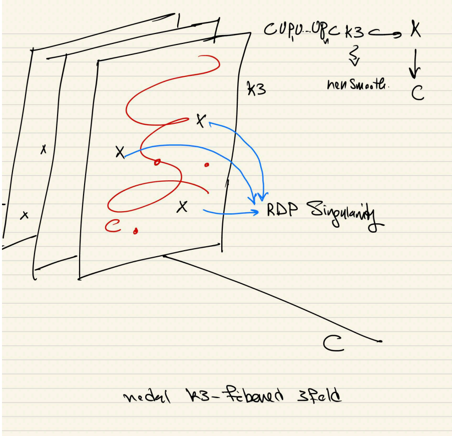
= (Göettche invariants → **Modular**) · (Noether-Lefschetz numbers → **Modular**; Borcherds)

$$Z(X,q) = \frac{\Phi^{\widetilde{\pi}}(q) - kv_0}{2\eta(q)^{24}},$$

where

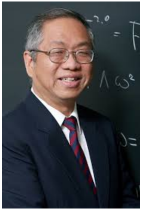
$$\Phi^{\widetilde{\pi}}(q) = \sum_{d=0}^{\ell-1} \Phi_d^{\widetilde{\pi}}(q) v_d \in \mathbb{C}[[q^{1/2\ell}]] \otimes \mathbb{C}[\mathbb{Z}/4\ell\mathbb{Z}]$$

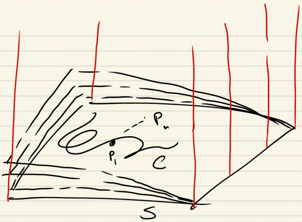
$$\Phi_d^{\widetilde{\pi}}(q) = q^{1+d^2/2\ell} \sum_{h\in\mathbb{Z}} NL_{h,d}^{\widetilde{\pi}} q^{-h},$$



Stable pairs on nodal K3 brations, (with Amin Gholampour and Yukinu Toda), International Mathematical Research Notices, Vol. 2017, No. 00, pp. 1-50, arXiv:1308.4722.

$$\sum_{h=0}^{\infty}\sum_{n=1-h}^{\infty}(-1)^{n+2h-1}\chi(P_n(S,h))y^n q^h =$$
$$-\left(\sqrt{-y}-\frac{1}{\sqrt{-y}}\right)^{-2}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{20}(1+yq^n)^2(1+y^{-1}q^n)^2}.$$





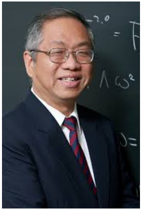
$X = \mathrm{tot}(\mathcal{L} \to S) \rightsquigarrow$ non compact.

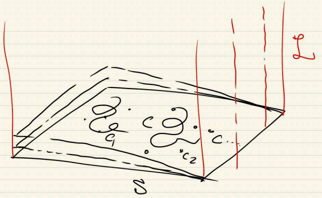
3 fold, line bdle., algebraic surface.

$C \cup P_1 \cup \cdots \cup P_n \subset S \subset X$

On topological approach to local theory of surfaces in Calabi-Yau threefolds, (with Sergei Gukov, Melissa Liu and Shing-Tung Yau), 39 pages, Advances in Theoretical and Mathematical Physics, Vol 21, no 7, p. 1679-1728 arXiv:609.04363.

$$\overline{\overline{\mathrm{DT}}}_h(v) = -\sum_{\substack{a_1+a_2=a \\ a_1h<a_2h}} \mathrm{SW}(a_1)\cdot 2^{1-\chi(v)}\cdot \mathcal{A}(a_1,a_2,v) + \widehat{\mathrm{DT}}_h(v).$$





$C_1 \cup P_1 \cup \cdots P_r \subset C_2 \cup P_1 \cup \cdots \cup P_r \subset \cdots \subset C_r \cup P_1 \cup \cdots P_L$

$\subset S \subset X$

$X = \mathrm{tot}(K_S \to S)$ noncomp

$\updownarrow$

Theory of Nested Hilbert Schemes

$\updownarrow$

!! Interestingly : Quantum Corrections

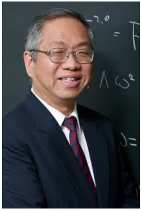
$\# VW = \# SW + \# \text{Nested}$

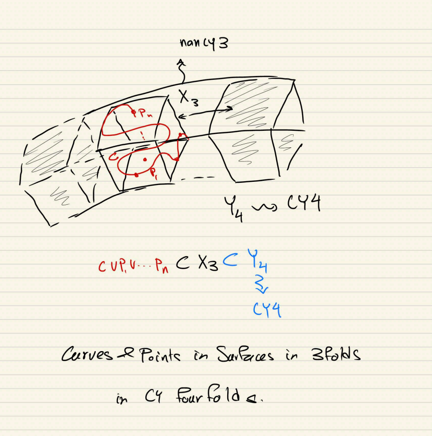
$\# VW \downarrow$ Vafa-Witten

$\# SW \downarrow$ Seiberg-Witten

Nested Hilbert schemes on surfaces: Virtual fundamental class, (with Amin Gholampour and Shing-Tung Yau), 47 pages, Advances in Mathematics, Vol 365, 13, May 2020 arXiv:1701.08899.

Localized Donaldson-Thomas theory of surfaces, (with Amin Gholmapour and Shing-Tung Yau), 28 pages, American Journal of Mathematics, Vol 142, 2, April 2020,





Atiyah class and sheaf counting on local Calabi Yau 4 folds, (with Emanuel Diaconescu and Shing-Tung Yau), Advances in Mathematics, Vol 368, 15 July, 2020, 54 pages, arXiv:1810.09382.

$$Z_{\infty}(X,2,f;q)_I = \frac{\mathbf{s}^{-1}}{2}\left(\Delta^{-1}(q^{1/2}) + \Delta^{-1}(-q^{1/2})\right), \tag{3}$$

where $\Delta(q)$ *is the discriminant modular form.*

Step 1 : Contribution of general rank 1 Sheaves with Support on hyperplane Section $H \subset X$.

Step ① : Contribution of general rank 1 Sheaves with Support on hyperplane Section $H \subset X$.

$$\begin{array}{ccc} & \text{Fano} & \text{Fano} \\ & \uparrow & \uparrow \end{array}$$

Strategy: degenerate $X \rightsquigarrow Y_1 \cup Y_2$

$$\begin{array}{cc} \wr\!\!\downarrow & \qquad\qquad S \\ CY3 & \end{array}$$

$S \subset Y_i$ anticanonical divisor.

$\mathbb{P}$: 4 dim'l, Smooth, Proj, Fano variety with ample anticanonical $K_{\mathbb{P}}^{\vee}$; $\mathbb{A}^1$: Spec $\mathbb{C}[t]$.

Consider trivial family $\pi_{\mathbb{P}} : \mathbb{P} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ & section

$$S \in H^0(\mathbb{P} \times \mathbb{A}^1, K_{\mathbb{P}}^{\vee} \boxtimes \mathcal{O}_{\mathbb{A}^1})$$

We view $S(t)$ as a Section of $K_{\mathbb{P}}^{\vee}$ dependjont.

fix a Splitting $\mathcal{K}_{\mathbb{P}}^{\vee} \simeq \mathcal{L}_1 \otimes \mathcal{L}_2$ $\quad \mathcal{L}_i$, $i=1,2$
ample line bundles

$\mathbb{P}$: "good degeneration" if

$t \neq 0$ $\quad Z(s(t)) =: X_t \subset \mathbb{P}$, Smooth CY3

$t \neq 0$ $\quad s(0) = s_1 \cdot s_2$; $s_i \in H^0(\mathbb{P}, \mathcal{L}_i)$ $i=1,2$

$Z(s_i) =: Y_i \subset \mathbb{P}$; 3 dim'l Fano

Y_1 & Y_2 intersect transversely along

their anticanonical divisor

Fix a Splitting $K_{\mathbb{P}}^{\vee} \simeq \mathcal{L}_1 \otimes \mathcal{L}_2$ $\quad \mathcal{L}_i$, $i=1,2$ ample line bundles

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$Z(s_i) =: Y_i \subset \mathbb{P}$; 3 dim'l Fano

Y_1 & Y_2 intersect transversely along their anticanonical divisor

Eg. $X_5 = Z(f_5(x_0 : \cdots : x_4)) \in \mathbb{P}^4$

$X_5 \rightsquigarrow Y_1 \underset{S}{\cup} Y_2$; $\quad t f_5(x_0 : \cdots : x_4) - f_1 f_4 = F(t)$

$Y_2 \cong \mathbb{P}3$

Y_1: Quartic $\subset \mathbb{P}4$

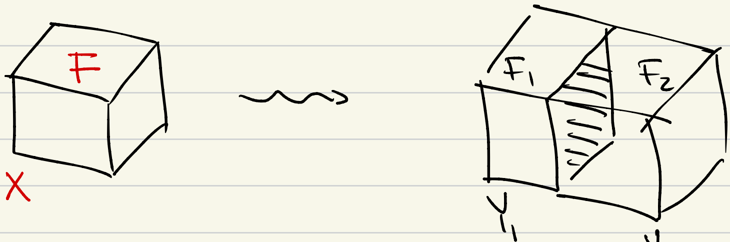
S: K3 Quartic

$t \neq 0 \rightsquigarrow Z(F(t)) =$ CY3 Smooth

$t = 0 \rightsquigarrow Z(F(0)) = Y_1 \underset{S}{\cup} Y_2$

Might need to Smoothen family $\quad X_5 \rightsquigarrow Y_1 \underset{S}{\cup} Y_2'$ e.g. $Bl_C \mathbb{P}^3$

Dream: Use degeneration technique in DT Theory.

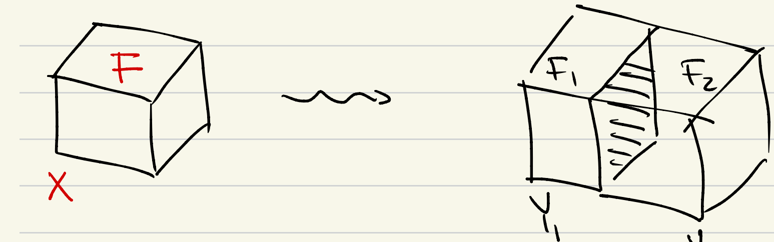


$\mathrm{DT}_{\bar{C}}(X)$ $\qquad$ $\mathrm{DT}_{\bar{C}_1}(Y_1/S)$ $\qquad$ $\mathrm{DT}_{\bar{C}_2}(Y_2/S)$

$$\mathrm{DT}_{\bar{C}}(X) = \sum_{\bar{C}=\bar{C}_1+\bar{C}_2} \mathrm{DT}(Y_1/S) \cdot \mathrm{DT}(Y_2/S)$$

Relative DT invts.

Dream: Use degeneration technique in DT theory.



$DT_{\bar{C}}(X)$ $\qquad$ $DT_{\bar{C}_1}(Y_1/S)$ $\qquad$ $DT_{\bar{C}_2}(Y_2/S)$

$$DT_{\bar{C}}(X) = \sum_{\bar{C}=\bar{C}_1+\bar{C}_2} DT(Y_1/S) \cdot DT(Y_2/S)$$

Relative DT invts.

$DT(Y_i/S)$ = DT invt of $\left(\begin{array}{l} F_i \in Coh(Y_i) \text{ which meet} \\ S \text{ (homologically) transversely} \\ \text{i.e. } Tor_1^{\mathcal{O}_{Y_i}}(F_i, \mathcal{O}_S) = 0 \end{array} \right)$

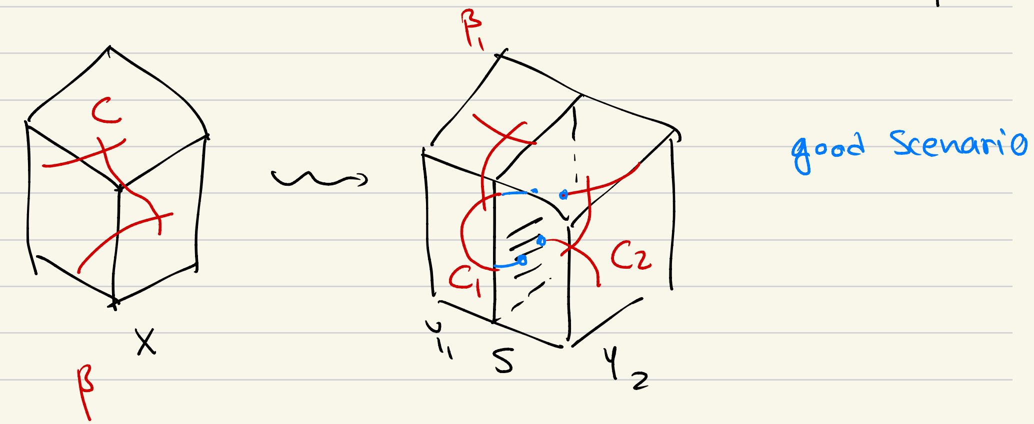
$i=1,2$

open condition

$$M_{\bar{C}_i}(Y_i/S) \subseteq M_{\bar{C}_i}(Y_i)$$

Jun Li (inventor of degeneration technique in GW thy)

Li-Wu (degeneration of ideal Sheaves & PT stable pairs)



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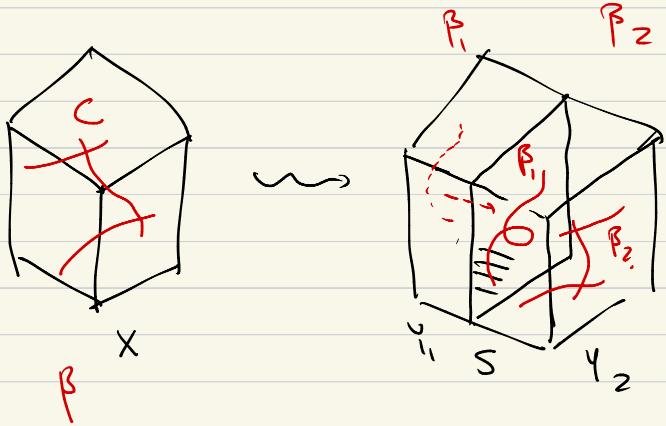
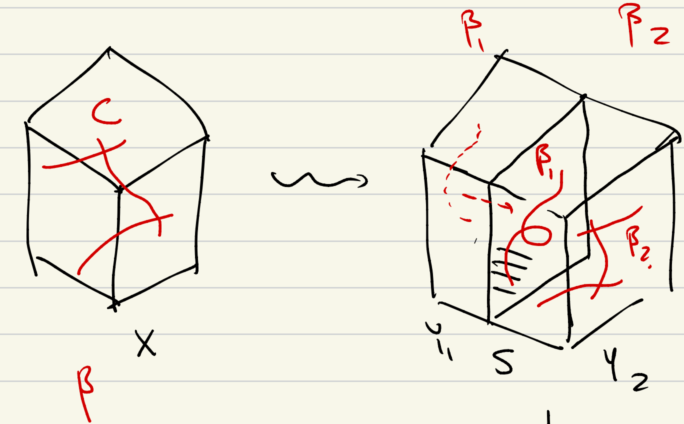


<u>BAD</u> Scenario

Jun Li (inventor of degeneration technique in GW thy)

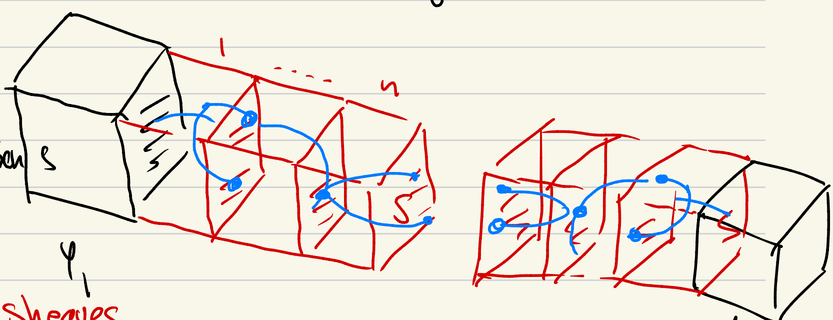
Li-Wu (degeneration of ideal Sheaves & PT stable pairs)



BAD Scenario

Replace with expanded degenerations

Expanded degenerations



Works for ideal sheaves & PT pairs

$Y_1[n]$ $Y_2[n]$ Y_2

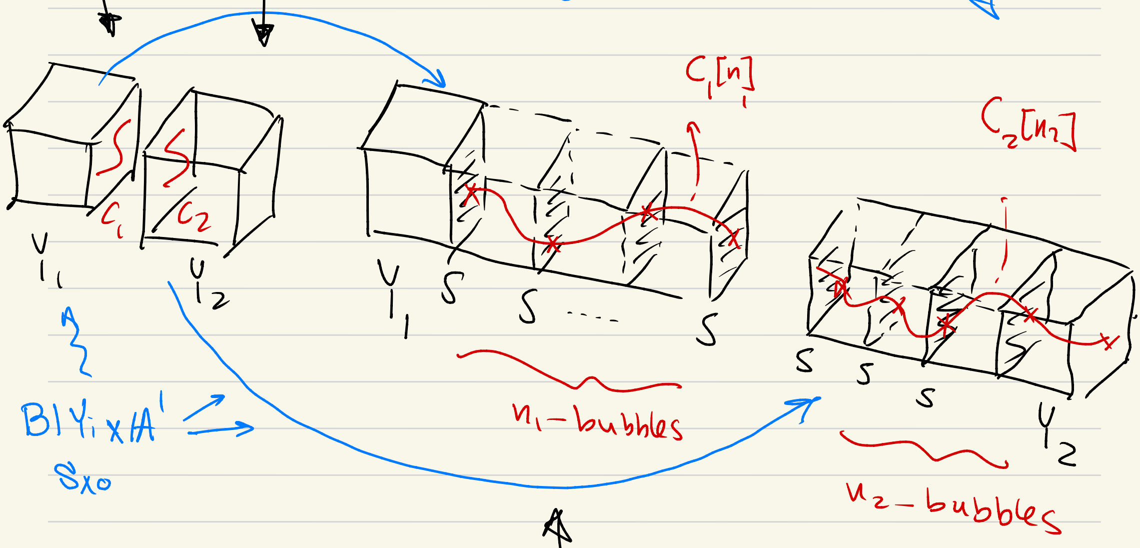
induces compactification of relative moduli sp $\overline{M(Y_i/S)}$

Li–Wu if F_i are ideal sheaves
or Pandharipande–Thomas
Stable pairs. (PT)

then expanded degenerations give Compactification

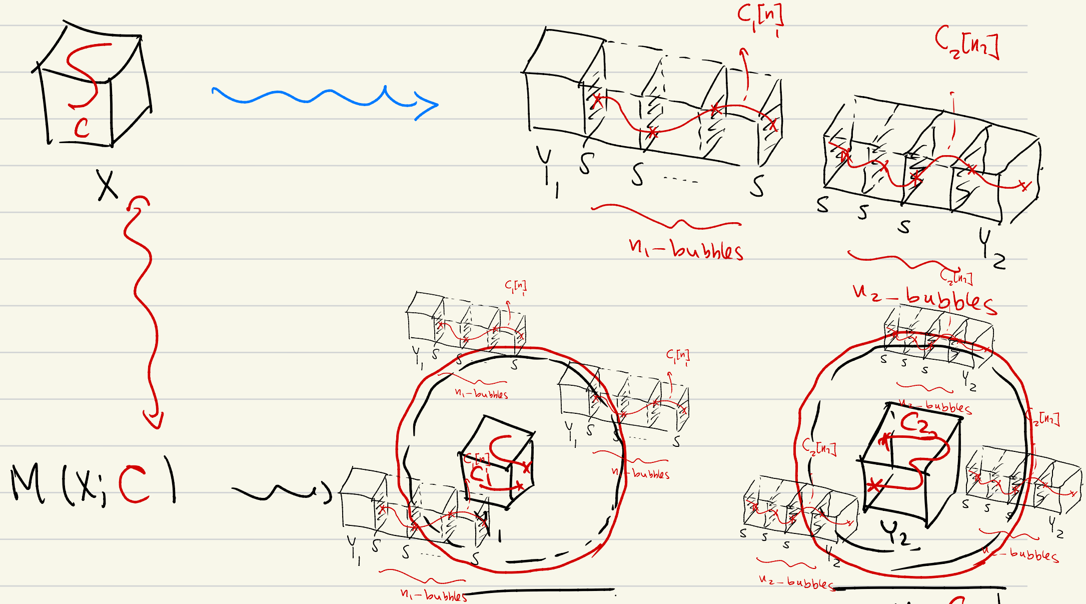
Bad Scenario

Similar to Jun Li's degeneration Program in GW theory.



Recall from GW Theory

in the limit!



$$\begin{tikzcd}
M(Y_1, C_1/S) \arrow[dr, "r_1"'] & & M(Y_2, C_2/S) \arrow[dl, "r_2"] \\
\mathrm{Hilb}^n(S) \arrow[r, "\Delta"] & \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^n(S) &
\end{tikzcd}$$

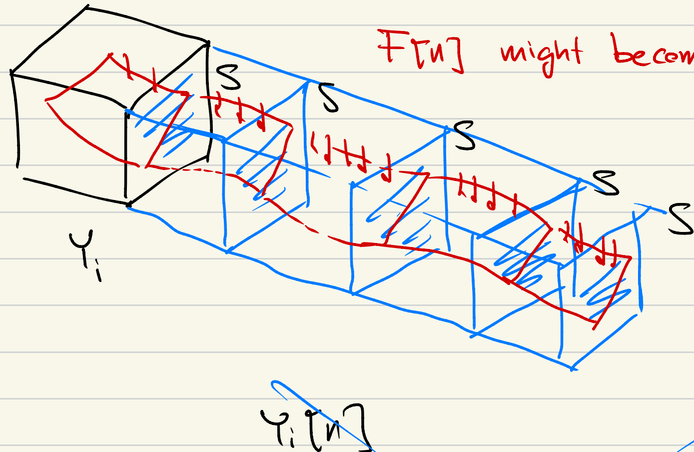
$$\Downarrow$$

$$[M(X;C)]^{vir} \underset{\text{invariance}}{\overset{\text{def'n}}{=}} \sum_{C = C_1 \cup C_2} \Delta^! \left([M(Y_1, C_1/S)]^{vir} \times [M(Y_2, C_2/S)]^{vir} \right)$$

Issue:

For general coherent sheaves Li_Wu compactification by expanded degenerations does not work!!!

$i=1,2$



$\overline{M}_{\bar{c}_i}(Y_i/S)$ is *not* well defined

Due to stability issues!

Remedy: Work with all sheaves (and the nontransverse ones)

and use derived intersection theory!

Perfect complexes

Remedy: Work with all sheaves (and the nontransverse ones) and use derived intersection theory!

ones $\updownarrow$ Perfect complexes

Dg-scheme of rigidified perfect complexes

$M(X) :=$ Moduli space of $F \in \mathrm{Perf}(X)$; which are rigidified

i.e. $\det(F) \cong \mathcal{L}_{fix} \in \mathrm{Pic}(X)$ and

① $\mathrm{Ext}^i(F,F) = 0 \quad i<0$

② trace map $\mathrm{Ext}^0_X(F,F) \to \mathbb{C}$ is an isom. (i.e. F is simple)

By (Schürg–Toën–Vezzosi 2015): The derived stack $M(X)$

is quasi-smooth (i.e. the def-obs complex $\dot{E}_{M(X)}$) is perfect of amplitude $[-1,0]$

define similarly $M(Y_1)$, $M(Y_2)$, $M(S)$ and assume they

are equipped with universal families $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_S$

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are equipped with universal families $\tilde{F}_1, \tilde{F}_2, \tilde{F}_S$

Since $S \underset{i=1,2}{\hookrightarrow} Y_i$ is a divisor $\Longrightarrow \exists$ natural (derived)

restriction map $\quad \tilde{F}_i \xrightarrow{r_i} \tilde{F}_i \overset{L}{\otimes} \mathcal{O}_{S \times M(S)} \quad i=1,2.$

define Similarly $M(Y_1)$, $M(Y_2)$, $M(S)$ and assume they are equipped with universal families $\tilde{F}_1, \tilde{F}_2, \tilde{F}_S$

Since $S \underset{i=1,2}{\hookrightarrow} Y_i$ is a divisor $\Longrightarrow$ $\exists$ natural (derived) restriction map $\tilde{F}_i \xrightarrow{r_i} \tilde{F}_i \overset{L}{\otimes} \mathcal{O}_{S \times M(S)}$ $i=1,2$.

$$\begin{tikzcd}
\underset{\bar{c}}{M(X)} \arrow[r, rightsquigarrow] & \underset{\bar{c}_1}{M(Y_1)} \arrow[dr, "r_1"] & & \underset{\bar{c}_2}{M(Y_2)} \arrow[dl] \\
& & \underset{\bar{c}_S}{M(S)} &
\end{tikzcd}$$

<u>Theorem</u> (Tyurin (Calac, BenBassat) ; (Dananevsky-Katzarkov-Kontsevich-S 2024)) Let $r_i : M(Y_i) \longrightarrow M(S)$ denote the derived restriction morphism. Then r_i Satisfies the Conditions of inducing a Lagrangian Structure.

i.e. $\exists$ induced map $\Theta_{r_i} = \mathbb{T}^{\bullet}_{r_i} \longrightarrow \mathbb{L}^{\bullet}_{M(Y_i)}[-1]$ is a quasi-isom of perfect cplxes.

Proof. $M(Y_i) \xrightarrow{f_i} M(S)$

exact Δ in $D^b(M(Y_i))$ $\quad \dot{\Pi}_{f_i} \longrightarrow \overset{\circ}{\Pi}_{M(Y_i)} \longrightarrow f_i^* \dot{\Pi}_{M(S)}$

Proof. $\quad M(Y_i) \xrightarrow{r_i} M(S)$

exact Δ in $D^b(M(Y_i))$ $\quad \dot{\mathbb{T}}_{r_i} \longrightarrow \dot{\mathbb{T}}_{M(Y_i)} \longrightarrow r_i^* \dot{\mathbb{T}}_{M(S)}$

$$\begin{array}{ccc} M(Y_i) \times Y_i & \xrightarrow{q_i} & Y_i \\ \downarrow{\scriptstyle p_i} & & \\ M(Y_i) & & \end{array}$$

Corresponds to

$$R\pi\, R\mathcal{H}om(\tilde{F}_i, \tilde{F}_i \otimes q_i^* \mathcal{O}_{Y_i}(-S))[1]$$

$$\downarrow$$

$$Rp_{i*}\, R\mathcal{H}om(\tilde{F}_i, \tilde{F}_i)[1]$$

$$\downarrow$$

$$Rp_{i*}(Ri_* Li^* R\mathcal{H}om(\tilde{F}_i, \tilde{F}_i))[1]$$

Proof. $\quad M(Y_i) \xrightarrow{r_i} M(S)$

exact Δ in $D^b(M(Y_i))$ $\quad \dot{\mathbb{T}}_{r_i} \longrightarrow \mathring{\mathbb{T}}_{M(Y_i)} \longrightarrow r_i^* \mathring{\mathbb{T}}_{M(S)}$

$$\begin{array}{ccc} M(Y_i)\times Y_i & \xrightarrow{q_i} & Y_i \\ \downarrow p_i & & \\ M(Y_i) & & \end{array}$$

Corresponds to

$$R_{\pi} R\mathcal{H}om(\check{F}_i, \check{F}_i \otimes q_{i}^* \mathcal{O}_{Y_i}(-S))[1]$$
$$\downarrow$$
$$Rp_{i*} R\mathcal{H}om(\tilde{F}_i, \tilde{F}_i)[1]$$
$$\downarrow$$
$$Rp_{i*}(Ri_*(i^* R\mathcal{H}om(\tilde{F}_i, \tilde{F}_i)))[1]$$

Inducing:

$$\begin{array}{ccc} \dot{\mathbb{T}}_{M(Y_i)} \otimes \mathring{\mathbb{T}}_{M(Y_i)} & \longrightarrow & Rp_{i*} \mathcal{O}_{Y_i \times M(Y_i)} \\ \downarrow & & \downarrow \\ r_i^* \dot{\mathbb{T}}_{M(S)} \otimes r_i^* \mathring{\mathbb{T}}_{M(S)} & \longrightarrow & r_i^* R^2 p_{i*} \mathcal{O}_{Y_i \times M(Y_i)} \end{array}$$

Proof. $M(Y_i) \xrightarrow{r_i} M(S)$

exact Δ in $D^b(M(Y_i))$: $\quad \dot{\mathbb{T}}_{r_i} \to \mathring{\mathbb{T}}_{M(Y_i)} \to r_i^* \mathring{\mathbb{T}}_{M(S)}$

Corresponds to

$$\begin{array}{ccc} M(Y_i)\times Y_i & \xrightarrow{q_i} & Y_i \\ \downarrow{\scriptstyle p_i} & & \\ M(Y_i) & & \end{array}$$

$$R_{\pi}\, R\mathcal{H}om\,(\check{\tilde{F}}_i, \check{\tilde{F}}_i \otimes q_{i}^{*}\, \mathcal{O}_{Y_i}(-S))[1]$$
$$\downarrow$$
$$Rp_{i*}\, R\mathcal{H}om\,(\tilde{F}_i, \tilde{F}_i)[1]$$
$$\downarrow$$
$$Rp_{i*}\,(Ri_* (i^* R\mathcal{H}om\,(\tilde{F}_i, \tilde{F}_i))[1]$$

Inducing:

Y_i Fano

$$\begin{array}{ccc} \dot{\mathbb{T}}_{M(Y_i)} \otimes \mathring{\mathbb{T}}_{M(Y_i)} & \longrightarrow & Rp_{i*}\, \mathcal{O}_{Y_i \times M(Y_i)} = 0 \\ \downarrow & & \downarrow \\ r_i^* \dot{\mathbb{T}}_{M(S)} \otimes r_i^* \mathring{\mathbb{T}}_{M(S)} & \longrightarrow & r_i^* R^2 p_{i*}\, \mathcal{O}_{Y_i \times M(Y_i)} \\ & & r_i^* \omega_{M(S)} \end{array}$$

$\Longrightarrow$ $\exists$ homotopy map between $r_i^* \omega_S$ and 0

Isotropic Structure!

For nondegeneracy need to show

$$\mathbb{T}^{-}_{M(\varphi_i)} \longrightarrow r_i^* \mathbb{T}^{\circ}_{M(S)} \xrightarrow{r_i^*\omega} r_i^* \mathbb{L}^{\circ}_{M(S)} \xrightarrow{-} \mathbb{L}^{\circ}_{M(\varphi_i)}$$

is q-isom to 0.

For nondegeneracy need to show

$$\mathbb{T}^\bullet_{M(\varphi_i)} \longrightarrow r_i^* \mathbb{T}^\bullet_{M(S)} \xrightarrow{r_i^*\omega} r_i^* \mathbb{L}^\bullet_{M(S)} \longrightarrow \mathbb{L}^\bullet_{M(\varphi_i)}$$

is q-isom to 0.

$$\cong$$

$$\begin{tikzcd}
\mathbb{T}^\bullet_{r_i} \arrow[r] & \mathbb{T}^\bullet_{M(\varphi_i)} \arrow[r] \arrow[d, "\alpha"] \arrow[dr, phantom, "?"] & r_i^* \mathbb{T}^\bullet_{M(S)} \arrow[d, "r_i^*\omega", "\cong"'] & \\
& \mathbb{L}^\bullet_{r_i}[-1] \arrow[r] & r_i^* \mathbb{L}^\bullet_{M(S)} \arrow[r] & \mathbb{L}^\bullet_{M(\varphi_i)}
\end{tikzcd}$$

q.e.d. $\square$

For nondegeneracy need to show

$$\mathbb{T}^{\circ}_{M(Y_i)} \to r_i^* \mathbb{T}^{\circ}_{M(S)} \xrightarrow{r_i^*\omega} r_i^* \mathbb{L}^{\circ}_{M(S)} \to \mathbb{L}^{\circ}_{M(Y_i)}$$

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$$\Vert$$

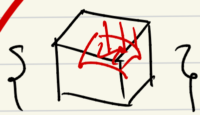
$$\begin{tikzcd}
\mathbb{T}^{\circ}_{r_i} \arrow[r] & \mathbb{T}^{\circ}_{M(Y_i)} \arrow[r] \arrow[d, "\alpha"] \arrow[dr, phantom, "?"] & r_i^* \mathbb{T}_{M(S)} \arrow[d, "\wr\, r_i^*\omega"] & \\
& \mathbb{L}^{\circ}_{r_i}[-1] \arrow[r] & r_i^* \mathbb{L}^{\circ}_{M(S)} \arrow[r] & \mathbb{L}^{\circ}_{M(Y_i)}
\end{tikzcd}$$

q.e.d. II

Corollary (PTVV): $M(Y_1) \underset{M(S)}{\times} M(Y_2)$ carries a (-1)-shifted symplectic structure, and by Joyce et al (BBBJ) $M(Y_1) \underset{M(S)}{\times} M(Y_2)$ is a derived critical locus of a function locally.

Categorified DT invts from derived Lagrangian intersection.

Need to show shifted symplectic structures are invt in degenerating family.



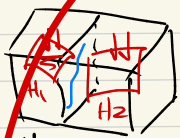
X

$\parallel$

$M(X, ch^*)$

$\parallel$

$d_{Dr}\,\mathrm{Crit}(f)^{t\neq 0}$



$Y_1 \qquad Y_2$

$M(Y_1, ch_1^*) \underset{M(S)}{\overset{DR}{\times}} M(Y_2, ch_2^\vee)$

$\parallel$

$d_{Dr}\,\mathrm{Crit}(f)^{t=0}$

deformation invariance of derived structure

$t\neq 0 \qquad t=0 \qquad \mathbb{A}^1$

let $\mathbb{P} = \mathrm{tot}\,(X \rightsquigarrow Y_1 \cup_S Y_2)$ Fano 4fold.

Need to Show $\rightsquigarrow$ All derived structure is induced from ambient space!!!

Derived Critical locus

Let f ; function on a Smooth scheme W

$d\,\mathrm{Crit}(f)$; represented by koszul algebra

$$(\mathrm{Sym}^* TW[1], df)$$

- The Cotangent Complex is a composition .

$$TW \xrightarrow{df} \mathcal{O}_W \xrightarrow{d_{dR}} \Omega_W$$

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Shifted Symplectic structure on W is reduced

to the Statement that

$$TW \longrightarrow \Omega_W$$

is self-dual!

Local model for dcrit locus on $M(X)$

Let X $\mathbb{CP}$ as before; $\mathcal{F} \in \mathrm{Coh}(X)$

Local model for dcrit locus on $M(X)$

Let $X \subset \mathbb{P}$ as before; $F \in Coh(X)$

Consider $\quad U = \mathrm{Ext}^1_X(F,F)$ $\qquad$ Serre $\qquad U^\vee = \mathrm{Ext}^2_X(F,F)$

$W_1 = \mathrm{Ext}^0_X(F, F\otimes K_{\mathbb{P}}^\vee)$ $\quad \xleftrightarrow{\text{duality}} \quad$ $W_1^\vee = \mathrm{Ext}^3_X(F\otimes K_{\mathbb{P}}^\vee, F)$

$W_2 = \mathrm{Ext}^1_X(F, F\otimes K_{\mathbb{P}}^\vee)$ $\qquad W_2^\vee = \mathrm{Ext}^2_X(F\otimes K_{\mathbb{P}}^\vee, F)$

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$\underrightarrow{\text{induces}}$ $\quad \mathrm{Ext}^\bullet$-algebra with L-∞ structure

on X
$$\begin{cases} L^1_X = U \\ L^2_X = U^\vee \\ L\text{-}\infty \text{ products } \quad l_k : \mathrm{Sym}^k(U) \longrightarrow U^\vee \end{cases}$$

Symmetric polys

taking adjoints

$$l_k^\vee : U \longrightarrow \mathrm{Sym}^k(U^\vee)$$

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Symmetric polys $\updownarrow$

$\updownarrow$ taking adjoints

$$l_k^\vee : \mathcal{U} \longrightarrow \mathrm{Sym}^k(\mathcal{U}^\vee)$$

$$\Longrightarrow \quad l^\vee_X := \sum_{k\geq 2} \frac{1}{k!}\left(\mathcal{U} \longrightarrow \mathrm{Sym}^k(\mathcal{U}^\vee)\right)$$

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Koszul Complex $\quad A^\circ_X = \left(\Lambda^\bullet(U[1]) \otimes \mathrm{Sym}^\bullet(U^\vee), d_{l^\vee_X}\right)$

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Let $X \subset \mathbb{P}$ as before; $F \in Coh(X)$

Consider $U = Ext^1_X(F,F)$ Serre duality $U^\vee = Ext^2_X(F,F)$

$W_1 = Ext^0_X(F, F \otimes K^\vee_{\mathbb{P}})$ $W_1^\vee = Ext^3_X(F \otimes K^\vee_{\mathbb{P}}, F)$

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$\xRightarrow{\text{induces}}$ $Ext^\bullet$-algebra with L-∞ structure

on X
$$\begin{cases} L^1_X = U \\ L^2_X = U^\vee \\ L\text{-}\infty \text{ products } \ell_k : Sym^k(U) \to U^\vee \end{cases}$$

Symmetric polys $\updownarrow$

$\updownarrow$ taking adjoints

dg-algebra of functions at a formal completion of $M(X)$ @ F.

$$\ell_k^\vee : U \to Sym^k(U^\vee)$$

$$\Longrightarrow \quad \ell_X^\vee := \sum_{k \geq 2} \frac{1}{k!} \left(U \to Sym^k(U^\vee) \right)$$

Koszul Complex $A^\circ_X = (\Lambda^\bullet(U[1]) \otimes Sym^\bullet(U^\vee), d_{\ell_X^\vee})$

Codim +1

Similar L-∞ algebra structure of $\mathcal{F} \in \mathrm{Coh}(\mathbb{P})$

$$A^{\bullet}_{\mathbb{P}} = \left(\mathrm{Sym}^{\bullet}(\mathcal{U}[1] \oplus W_2^{\vee}[1]) \otimes \mathrm{Sym}^{\bullet}(\mathcal{U}^{\vee} \oplus W_1^{\vee}), d_{e_{\mathbb{P}}^{\vee}}\right)$$

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Lemma (BKKS, 2024) Let $f := \sum_{k \geq 2} \frac{1}{(k+1)!} f_{k+1} \in \mathrm{Sym}^*(\mathcal{U}^\vee)$

then dg-algebra $A^\bullet_X \cong K^\bullet(df)$

$K^\bullet$: koszul algebra

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$K^\bullet$ $\downarrow$ Koszul algebra

Remark This property **Fails!** for $A^\bullet_{\mathbb{P}}$ since $\mathbb{P}$ is not CY. and $\mathbb{L}_{\mathbb{P}}$ doesnt have shifted self-duality.

However one can recover A°_X from terms in $A^\circ_{\mathbb{P}}$

Introduce new L-∞ algebra structure $L^{\bullet}_{+}$

$$L^1_+ = \mathcal{U} \oplus W_2 \oplus W_2^\vee = \mathrm{Ext}^1_X(F,F) \oplus \mathrm{Ext}^0(F, F\otimes K^\vee_{\mathbb{P}}) \oplus \mathrm{Ext}^2_X(F\otimes K^\vee_{\mathbb{P}}, F)$$

$$L^2_+ = \mathcal{U}^\vee \oplus W_2 \oplus W_1^\vee = \mathrm{Ext}^2_X(F,F) \oplus \mathrm{Ext}^1(F, F\otimes K^\vee_{\mathbb{P}}) \oplus \mathrm{Ext}^3_X(F\otimes K^\vee_{\mathbb{P}}, F)$$

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$$\Big\downarrow \text{ Posseses } L_\infty \text{ products}$$

1. $\mathrm{Sym}^k(\mathcal{U}) \to \mathcal{U}^\vee$

2. $\mathrm{Sym}^{k-1}(\mathcal{U}) \otimes W_1 \to W_2$

3. $\mathrm{Sym}^{k-1}(\mathcal{U}) \otimes W_2^\vee \longrightarrow W_1^\vee$

4. $\mathrm{Sym}^{k-2}(\mathcal{U}) \otimes W_2^\vee \otimes W_1 \longrightarrow \mathcal{U}^\vee$ $\quad k \geq 2$

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$\downarrow$ Posseses L_∞ products

① $\mathrm{Sym}^k(\mathcal{U}) \to \mathcal{U}^\vee \longrightarrow \mathrm{Ext}^{\bullet}_X(F,F)$

② $\mathrm{Sym}^{k-1}(\mathcal{U}) \otimes W_1 \to W_2$

③ $\mathrm{Sym}^{k-1}(\mathcal{U}) \otimes W_2^\vee \longrightarrow W_1^\vee$

④ $\mathrm{Sym}^{k-2}(\mathcal{U}) \otimes W_2^\vee \otimes W_1 \longrightarrow \mathcal{U}^\vee$ $\quad k \geq 2$

Partial derivatives

$$\mathcal{U} \otimes W_1 \otimes W_2^\vee \xrightarrow{g} \mathbb{C}$$

Introduce new L-∞ algebra structure $L_+^{\bullet}$

$$L_+^1 = \mathcal{U} \oplus W_2 \oplus W_2^\vee = \mathrm{Ext}_X^1(F,F) \oplus \mathrm{Ext}^0(F, F\otimes k_{\mathbb{P}}^\vee) \oplus \mathrm{Ext}_X^2(F\otimes k_{\mathbb{P}}^\vee, F)$$

$$L_+^2 = \mathcal{U}^\vee \oplus W_2 \oplus W_1^\vee = \mathrm{Ext}_X^2(F,F) \oplus \mathrm{Ext}^1(F, F\otimes k_{\mathbb{P}}^\vee) \oplus \mathrm{Ext}_X^3(F\otimes k_{\mathbb{P}}^\vee, F)$$

$\big\downarrow$ Posseses L_∞ products

1. $\mathrm{Sym}^k(\mathcal{U}) \to \mathcal{U}^\vee \longrightarrow \mathrm{Ext}_X^{\bullet}(F,F)$

2. $\mathrm{Sym}^{k-1}(\mathcal{U}) \otimes W_1 \to W_2$

3. $\mathrm{Sym}^{k-1}(\mathcal{U}) \otimes W_2^\vee \longrightarrow W_1^\vee$

4. $\mathrm{Sym}^{k-2}(\mathcal{U}) \otimes W_2^\vee \otimes W_1 \longrightarrow \mathcal{U}^\vee \quad k \geq 2$

Partial derivatives

$$\mathcal{U} \otimes W_1 \otimes W_2^\vee \xrightarrow{g} \mathbb{C}$$

<u>Lemma</u> The products $\mathrm{Sym}^k(\mathcal{U}) \to \mathcal{U}^\vee$, $\mathrm{Sym}^{k-1}(\mathcal{U}) \otimes W_1 \xrightarrow{k \geq 2} W_2$

Can be chosen to describe the Canonical L-∞ algebra

Structure $\mathrm{Ext}^{\bullet}_{\mathbb{P}}(F,F) \Longrightarrow$ Existence of g defines the two Series of products uniquely.

<u>Pf</u> Uses Kodaira vanishing and that

$$H^i(X, \mathcal{H}om(\mathrm{Sym}^\alpha K^\vee_{\mathbb{P}}|_X, K^\vee_{\mathbb{P}}|_X)) = 0 \ \forall i>0$$

$\rightsquigarrow$

Replace $\mathbb{P}$ by total space of normal bundle

$\mathbb{P}' := K^\vee_{\mathbb{P}}|_X \to X$

$\rightsquigarrow$

any sheaf $\mathcal{F}$ can be viewed as

derived restriction of its pull back p^* to

$K^\vee_{\mathbb{P}}|_X \to X$.

$\rightsquigarrow$

$\mathrm{Ext}^\bullet_X(\mathcal{F}, \mathcal{E})$ and $\mathrm{Ext}^\bullet_{\mathbb{P}'}(\mathcal{F}, \mathcal{E})$ are viewed

as cohomology of $R\mathcal{H}om_{\mathbb{P}'}(p^*, p^*)$ and

replace with Koszul resoln

$R\mathcal{H}om_{\mathbb{P}'}(p^* \otimes \mathcal{O}_X, p^* \otimes \mathcal{O}_X)$

Cor The dg-algebra

$$A^{\bullet +}_{\mathbb{P}} = \mathrm{Sym}^{\bullet}(\mathcal{U}[1] \oplus W_2^{\vee}[1] \oplus W_1[1]) \otimes \mathrm{Sym}^{\bullet}(\mathcal{U}^{\vee} \oplus W_1^{\vee} \oplus W_2)$$

is a Symmetric algebra of dg-module

$$W_1[1] \otimes A^{\circ}_{\mathbb{P}} \longrightarrow W_2 \otimes A^{\circ}_{\mathbb{P}}$$

Cor The dg-algebra

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is a Symmetric algebra of dg-module

$$W_1[1] \otimes A^{\circ}_{\mathbb{P}} \longrightarrow W_2 \otimes A^{\circ}_{\mathbb{P}}$$

with differential obtained from (linear) map

$$W_1 \longrightarrow W_2 \otimes \mathrm{Sym}^{*}(\mathcal{U}^{\vee})$$

Induced by g!

Cor 2 (BKKS 2024) The dg-algebra $A^{\bullet}_{X} \overset{q\text{-isom}}{\simeq}$ to

dg-algebra of $\mathrm{dCrit}\,(f+g)$ on $\mathcal{U} \oplus W_2^{\vee} \oplus W_1$

Quasi-BPS Categories for families of potentials and the Gauss-Manin Connection on Periodic Cyclic homology.

Consider $M_{\bar{c}}(\mathbb{P})$ the quasi-Smooth Scheme

and Consider Family of potentials Constructed in

previous step $\mathcal{T}^c(s) = f(s) + g(s)$ depending on the

Choice of $s \in H^0(\mathbb{P}, K_{\mathbb{P}}^{\vee})$

Quasi-BPS Categories for families of potentials and the Gauss-Manin Connection on Periodic Cyclic homology.

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Padurario-Toda: quasi-BPS category $\mathcal{D}(s)$ of $M(X(s))$

As a $\mathbb{Z}_2$-graded Category of Matrix-Factorizations

of $\mathbb{K}(s)$.

As a $\mathbb{Z}_2$-graded Category of Matrix-factorizations

of $\Gamma(s)$ $\overset{\text{i.e.}}{:\Rightarrow}$ $D(s)$ = dg-category of $\mathbb{Z}_2$-graded

loc free sheaves $E = E^0 \oplus E^1$ on $M(X(s))$ with

an odd diff'l δ which satisfies

$$\delta^2 = \Gamma(s) \cdot Id$$

With complexes of morphisms defined by taking

Standard Hom-Complexes

$\{$set of morphisms in $D(s)\}$ = $\mathbb{Z}_2$-graded Vector Spaces with

odd differential d and even Curvature elements

$h_X \in \mathrm{Hom}_{D(s)}(X, X)$ $\forall X \in D(s)$

① d satisfies Leibniz rule w.r.t Composition morphism

② for $f \in \mathrm{Hom}_{D(s)}(X, Y)$.

$d^2(f) = h_Y f - f h_X$

③ $\forall X$, $dh_X = 0$

④ $\forall X$, id_X has degree 0.

$d\, id_X = 0$ $\leftarrow$

PT $\overset{\text{defin}}{\rightsquigarrow}$ Hochschild homology of $D(s)$

$\updownarrow$

Matrix factorization category

PT $\overset{\text{defin}}{\rightsquigarrow}$ Hochschild homology of $D(S)$

$\Updownarrow$

Matrix factorization category

induces $\Downarrow$

Periodic Cyclic homology $HP(S)$ of $D(S)$

$$\left(H^*\left(\mathrm{Hoch}^\bullet(D(S))((u)), \underbrace{b+uB}_{\text{diffl}} \right)\right)$$

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Periodic Cyclic homology $HP(S)$ of $D(S)$

$\left(H^*\left(\mathrm{Hoch}^\bullet(D(S))((u)), \underbrace{b+uB}_{\text{diff'l}} \right)\right)$

$\Downarrow$ Efimov 2018

$HP^*(S) \underset{\mathbb{Z}_2\text{-graded}}{\cong}$ Cohomology (in Zariski topology)

of the Complex $\left(\underbrace{\Omega^\bullet_{M(S)}((u))}_{\text{deRham Cplx.}}, -d\Gamma_{(S)} + u\, d_{dR} \right)$

Thm (BKKS 2024) For a good degeneration

$$\mathbb{P}: X \rightsquigarrow Y_1 \underset{S}{\cup} Y_2$$

$\exists$ a flat Connection on a vector bundle over $\mathbb{A}^1$

with fiber $HP\Big(\mathcal{D}(s) \text{ of } \begin{array}{c} M(s) \\ \downarrow f \\ \mathbb{C} \end{array}\Big)$. In particular

the graded dimension of $HP(\mathcal{D}(s))$ is Constant

in family.

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Remark In the setting where $\mathrm{dcrit}(\Gamma(s)) \subset \underset{\substack{\Downarrow \\ \text{vanishing locus}}}{Z(\Gamma(s))}$ the twisted de Rham complex in above is q-isomorphic to the sheaf of vanishing cycles on $\Gamma(s)$. $\rightsquigarrow$ its cohomology gives categorification of DT invts. $\rightsquigarrow$ $\mathcal{D}(s)$ can be regarded as higher level categorification!

Computation of DT cohomology groups over the special fiber.

Computation of DT cohomology groups over the special fiber.

Given Lagrangians L and M in hol. Sympl mfd S

with choice of $K_L^{\frac{1}{2}}$ and $K_M^{\frac{1}{2}}$

Thm (Cunningham – Safronov)
2023

$$L \otimes^{\mathbb{L}}_{P(S)} M \simeq \phi_{L \cap M} \otimes_{\mathbb{C}} \mathbb{C}(\hbar)$$

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Thm (Cunningham – Safronov 2023)

$$\underbrace{L \otimes^{\mathbb{L}}_{P(S)} M}_{} \cong \underbrace{\phi_{L \cap M} \otimes_{\mathbb{C}} \mathbb{C}(\hbar)}_{}$$

quantization of modules over $\mathcal{P}_S = P_{S(0)} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar))$

Perverse Sheaf of vanishing Cycles associated to Lagrangian intersection.

Our approach : Similar derived geometricall

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Let $K^{\frac{1}{2}}_{M(Y_1)}$ and $K^{\frac{1}{2}}_{M(Y_2)}$

$$\begin{tikzcd}
K^{\frac{1}{2}}_{M(Y_1)} \arrow[d, leftrightarrow] & K^{\frac{1}{2}}_{M(Y_2)} \arrow[d, leftrightarrow] \\
\det\left(\mathbb{E}^{\bullet}_{M(Y_1)}\right)^{\frac{1}{2}} \arrow[d, squiggly, "\substack{\text{deformation quantization}\\ \text{to left } \mathbb{C}[[\hbar]]\text{-flat module}}"'] & \det\left(\mathbb{E}^{\circ}_{M(Y_2)}\right)^{\frac{1}{2}} \arrow[d, squiggly, "\substack{\text{deformation quantization}\\ \text{to left } \mathbb{C}[[\hbar]]\text{-flat module}}"] \\
\mathcal{L}_{M(Y_1)} & \mathcal{M}_{M(Y_2)}
\end{tikzcd}$$

Our approach: Similar derived geometrically

let $K^{\frac{1}{2}}_{M(Y_1)}$ and $K^{\frac{1}{2}}_{M(Y_2)}$.

$$\det\left(\mathbb{E}^\bullet_{M(Y_1)}\right)^{\frac{1}{2}} \qquad \det\left(\mathbb{E}^\bullet_{M(Y_2)}\right)^{\frac{1}{2}}$$

deformation quantization to left $\mathbb{C}[[\hbar]]$-flat module $K^{\frac{1}{2},\hbar}_{M(Y_1)}$

deformation quantization to left $\mathbb{C}[[\hbar]]$-flat module $K^{\frac{1}{2},\hbar}_{M(Y_2)}$

$$K^{\frac{1}{2},\hbar^\vee}_{M(Y_1)} := R\mathcal{H}om_{\mathcal{O}_\hbar}\left(K^{\frac{1}{2},\hbar}_{M(Y_1)}, \mathcal{O}_\hbar\right)\left[\frac{\dim^{vir}(M(S))}{2}\right]$$

$$\Downarrow$$

$$\underbrace{K^{\frac{1}{2},\hbar^\vee}_{M(Y_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\hbar} K^{\frac{1}{2},\hbar}_{M(Y_2)}}_{\text{Use Bar resol'n}} \underset{q\text{-isom}}{\simeq} \underbrace{\mathcal{O}^{op}_\hbar \otimes^{\mathbb{L}}_{\mathcal{O}^e_\hbar}\left(K^{\frac{1}{2},\hbar}_{M(Y_2)} \otimes_{\mathbb{C}[[\hbar]]} K^{\frac{1}{2},\hbar}_{M(Y_1)}\right)}_{\text{Use Bar resol'n}}$$

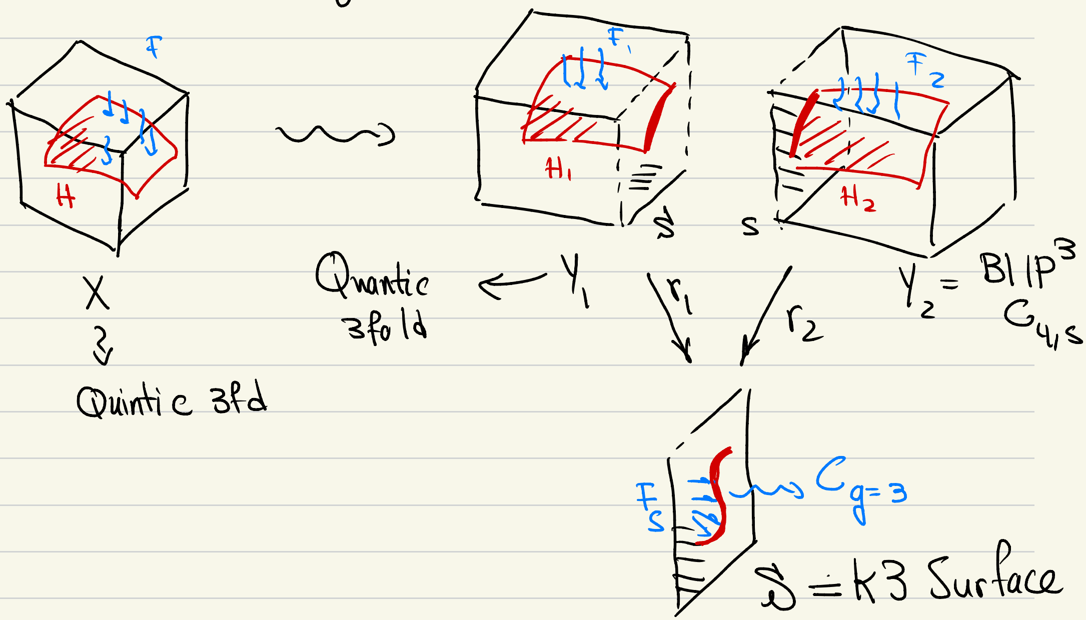
Conjecture (BKKS 2024 - partially work in progress)

$$\sum_{\bar{c}=\bar{c}_1+\bar{c}_2} \chi\left(H^{\bullet}\left(\mathcal{K}^{\frac{1}{2},\hbar^\vee}_{M(Y_1)_{\bar{c}}} \overset{L}{\underset{\mathcal{O}_\hbar}{\otimes}} \mathcal{K}^{\frac{1}{2},\hbar}_{M(Y_2)_{\bar{c}_2}} \right)\right)$$

$$= \sum_{\bar{c}=\bar{c}_1+\bar{c}_2} DT_{\bar{c}_1}(Y_1/S) \cdot DT_{\bar{c}_2}(Y_2/S)$$

Generic Picture & S-duality Conjecture

Eg: Quintic 3fold degeneration



Generic Picture & S-duality Conjecture

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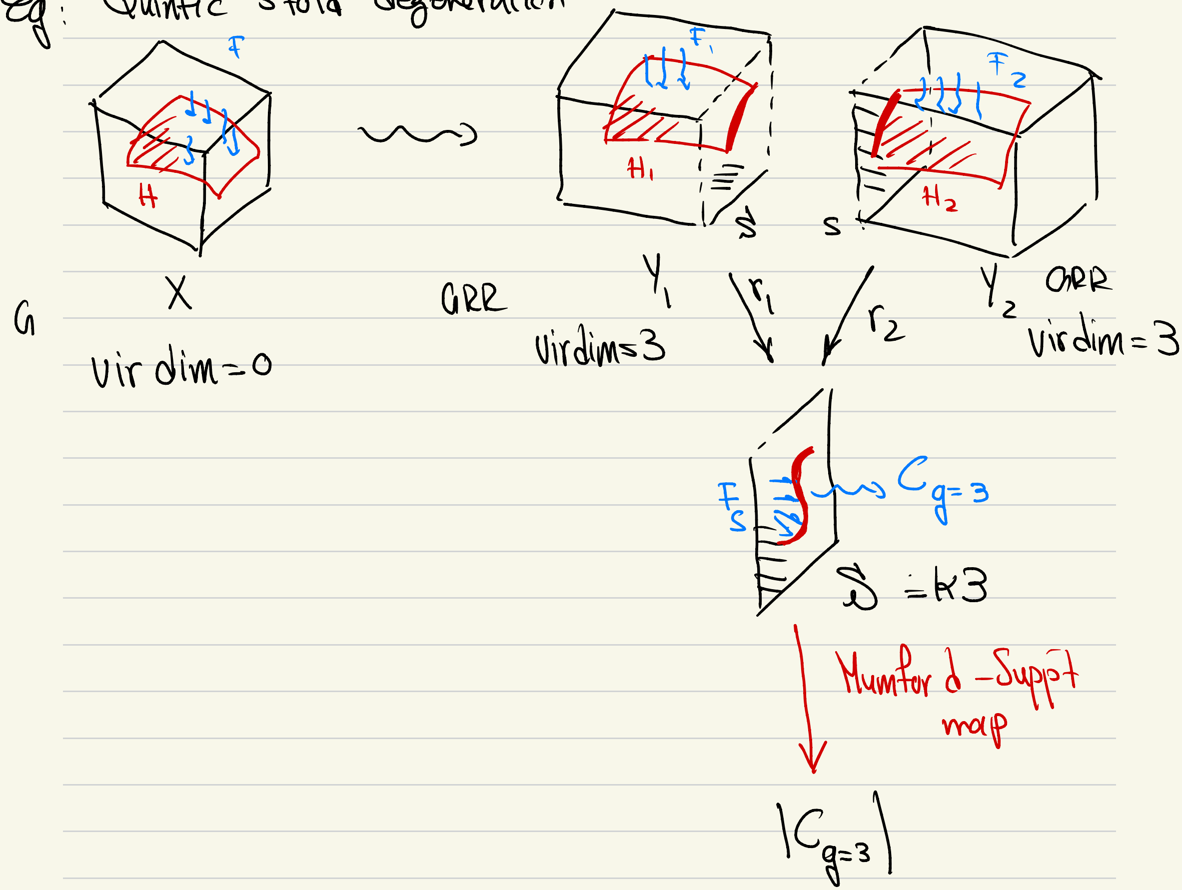

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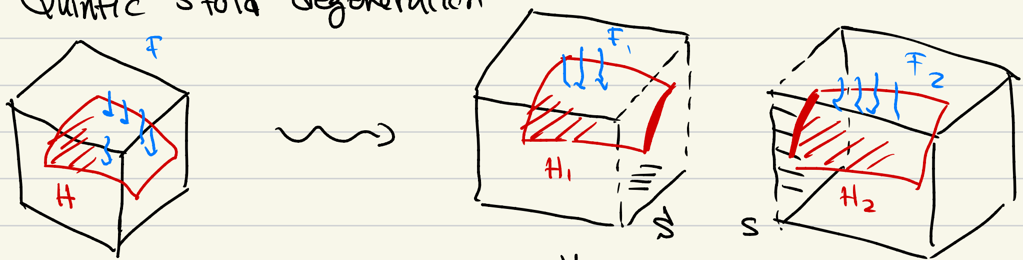


$X \rightsquigarrow Y_1 \quad Y_2$

X: GRR, vir dim $=0$

Y_1: GRR, virdim $=3$

Y_2: GRR, virdim $=3$

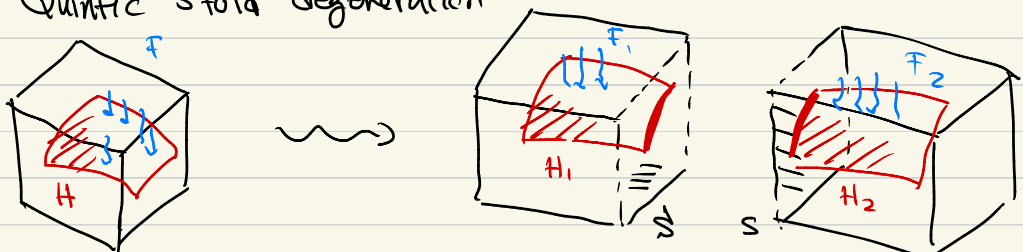
$$Y_1 \xrightarrow{r_1} \overline{\mathrm{Jac}}\left(\mathcal{C}_{g=3} / |C_{g=3}|\right) \xleftarrow{r_2} Y_2$$

$\overline{\mathrm{Jac}}\left(\mathcal{C}_{g=3} / |C_{g=3}|\right)$: dim $=6$ & Smooth; hol. Symp mfd

$$\overline{\mathrm{Jac}}\left(\mathcal{C}_{g=3} / |C_{g=3}|\right) \xrightarrow{\text{Mumford-Supp't map}} |C_{g=3}| = \mathbb{P}^3$$

Generic Picture & S-duality Conjecture

Eg: Quintic 3fold degeneration



X GRR vir dim = 0

GRR Y_1 virdim = 3 $\qquad$ Y_2 GRR virdim = 3

$$
\begin{tikzcd}
Y_1 \arrow[dr, "r_1"] & & Y_2 \arrow[dl, "r_2"'] \\
& \overline{\mathrm{Jac}}\left(C_{g=3} / |C_{g=3}|\right) \arrow[d, "\pi"] & \\
& |C_{g=3}| &
\end{tikzcd}
$$

dim = 6 & Smooth $\longleftarrow$ $\overline{\mathrm{Jac}}\left(C_{g=3} / |C_{g=3}|\right)$ hol. Symp mfd

Lagrangian fibration with a Section

$\ni C_{g=3} \rightsquigarrow pt \in |C_{g=3}| = \mathbb{P}^3$

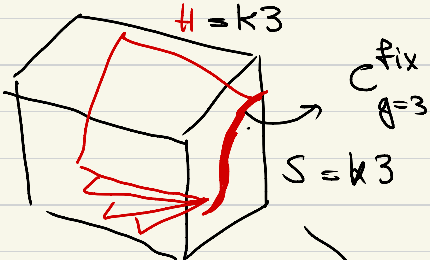
$$\overline{DT}_{\tau_1}(Y_1/S) = \overline{DT}_{\tau_1}(Y_1, \underbrace{\alpha}_{\text{insertion Cohomology class}}) = \int_{[M(Y_i)]^{vir}} \overbrace{(r_i \circ \pi)^* [pt]}^{\alpha}$$

By localization:

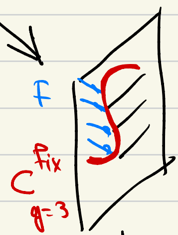
$$\int_{[\overline{M}(Y_i)]^{vir}} (r_i \circ \pi)^* [pt] = \int_{[\underbrace{(\pi \circ r_i)^*(pt)}_{\text{Sublocus}} \subset \overline{M}(Y_i)]^{vir}} 1 .$$

By localization:

$$\int_{[M(Y_i)_{\bar{C}_i}]^{vir}} (r_i \circ \pi)^* [pt] = \int_{[\underbrace{(\pi \circ r_i)^*(pt)}_{\text{Sublocus}} \subset M(Y_i)_{\bar{C}_i}]^{vir}} 1 .$$



Quantic 3fold = Y_1

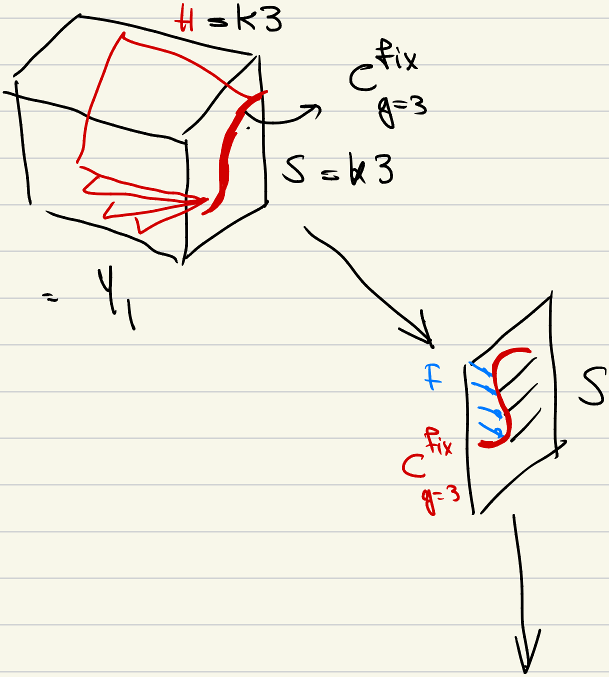


$S \in \overline{Jac}(C^{fix}_{g=3}) \subset \overline{Jac}\, C_{g=3} / |C_{g=3}|$

$$C^{fix}_{g=3} = pt \in \mathbb{P}^3 = |C_{g=3}|$$

By localization:

$$\int_{[M(Y_i)_{\bar{C}_i}]^{vir}} (r_i \circ \pi)^* [pt] = \int_{[\underbrace{(\pi \circ r_i)^*(pt)}_{\text{Sublocus}} \subset M(Y_i)_{\bar{C}_i}]^{vir}} 1 .$$



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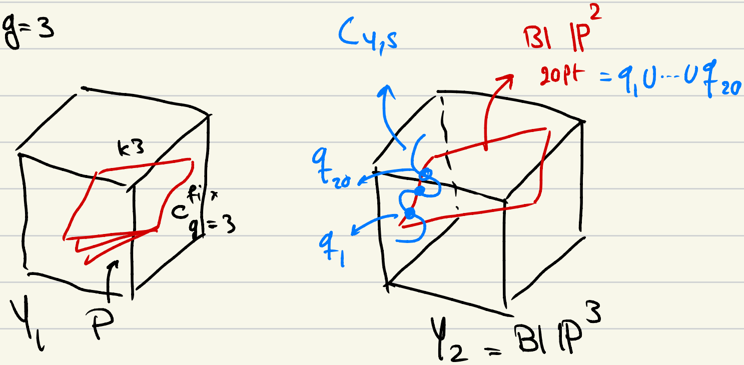
$DT(Y_1/S, \alpha) \rightsquigarrow$ Contribution of pencil of

Quartic K3's to $DT(Y_1)$ with base locus

$C^{fix}_{g=3}$



$DT(Y_2/S, \alpha) \rightsquigarrow$ Contribution of $\mathrm{Hilb}_n(Bl\, \mathbb{P}^2_{20pt} / C^{fix}_{g=3})$

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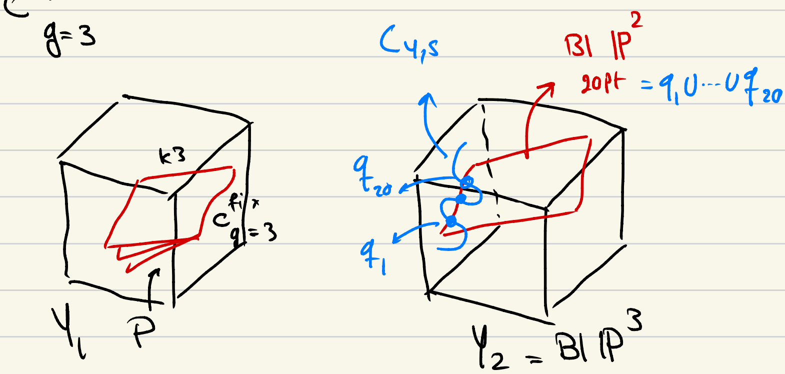

$DT(Y_2/S, \alpha) \rightsquigarrow$ Contribution of $\mathrm{Hilb}_n(Bl_{20pt}\mathbb{P}^2 / C^{fix}_{g=3})$

$$DT_{\bar{C}}(X) = \sum_{\bar{c} = \bar{c}_1 + \bar{c}_2} DT(Y_1/S, \alpha) \cdot DT(\mathrm{Hilb}(Bl_{20pt}\mathbb{P}^2 / C^{fix}_{g=3}))$$

$$Z(DT_{\bar{c}}(X)) = Z(DT(Y_1/S, \alpha)) \cdot Z(DT(\text{"} \quad \text{"}))$$

modular?

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modular form.

Let $\tilde{Y}_1 = \mathrm{Bl}_{C^{\mathrm{fix}}_{g=3}} Y_1 \implies \check{Y}_1 =$ K3-fibered 3fold over $C^{\mathrm{fix}}_{C_{g=3}}$

Relate $DT(\check{Y}_1)$ and $DT(Y_1/_S\,|\alpha)$ via degeneration

Let $Y_1 \rightsquigarrow Y_1 \cup \mathbb{P}(\mathcal{O}_S \oplus N^\vee_{S/Y_1})$ degeneration to normal cone of $S \subset Y_1$

Let $\tilde{Y}_1 = \mathrm{Bl}_{C^{fix}_{g=3}} Y_1 \implies \tilde{Y}_1 =$ K3-fibered 3fold over $C^{fix}_{C_{g=3}}$

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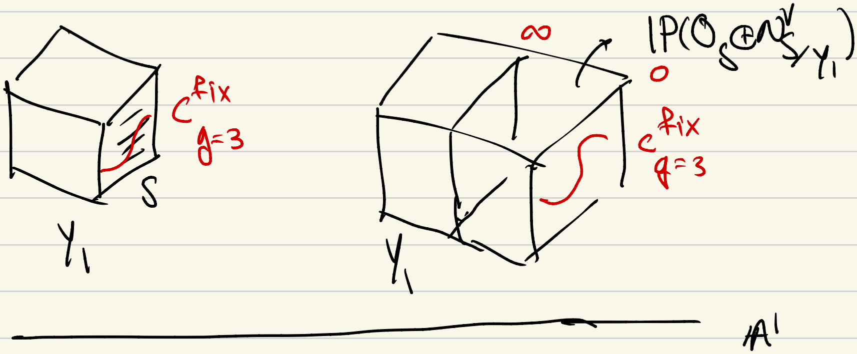

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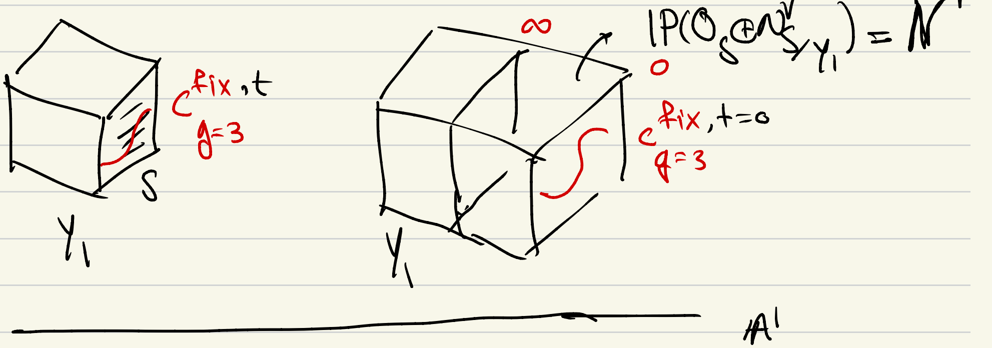
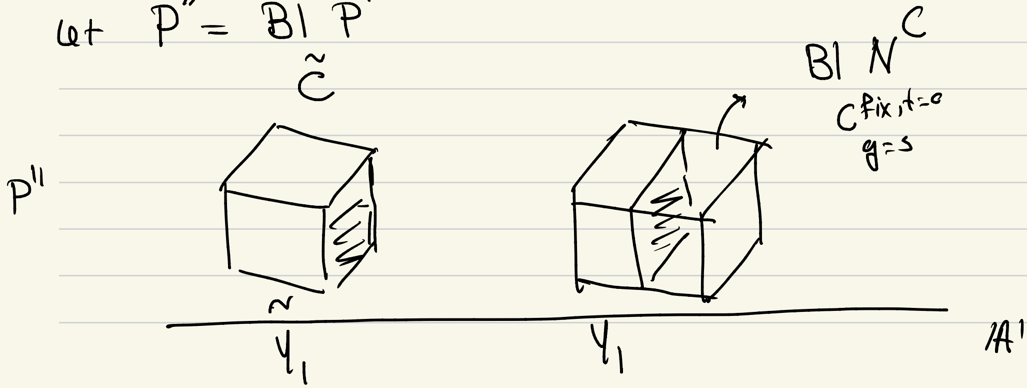


Let $\tilde{C} = \lim_{t \to 0} C^{fix,t}_{g=3}$

Let $P'' = Bl_{\tilde{C}} P'$



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Relate $DT(\tilde{Y}_1)$ and $DT(Y_1/_S, \alpha)$ via degeneration

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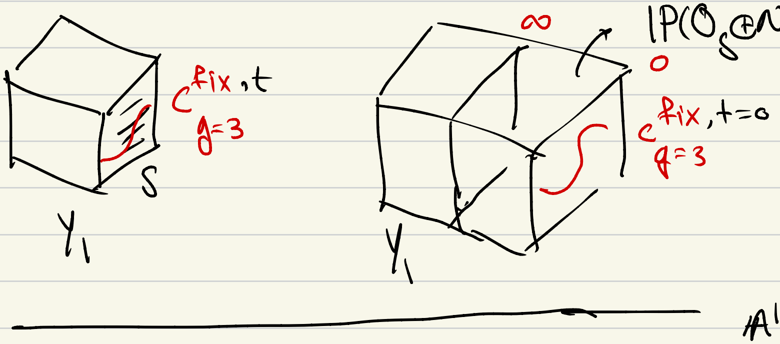
$\mathbb{P}(\mathcal{O}_S \oplus N^\vee_{S/Y_1}) = N^\vee$

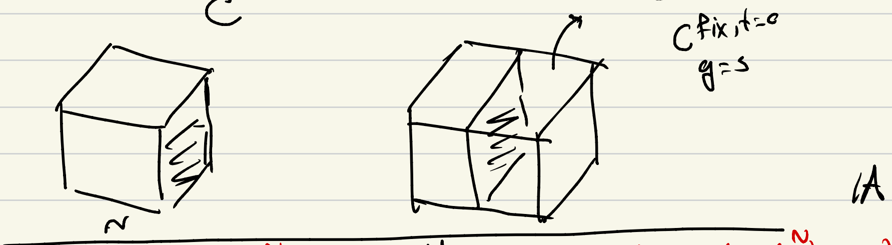


Let $\tilde{C} = \lim_{t \to 0} C^{fix,t}_{g=3}$

Let $P'' = \mathrm{Bl}_{\tilde{C}} P'$

$\mathrm{Bl}_{C^{fix,t=0}_{g=3}} N^C =: \tilde{N}^C$

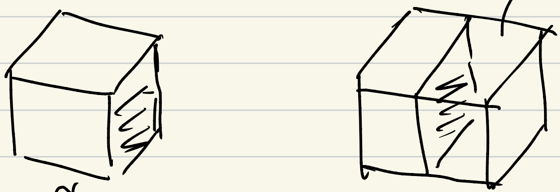
P''



$\tilde{Y}_1$ $Z(DT(\tilde{Y}_1)) = Y_1$ $Z(DT(Y_1/_S)) \cdot Z(DT(\tilde{N}_C/_S))$

$$\mathrm{Bl}\, N^{C} =: \tilde{N}^{C}$$

$$C^{\mathrm{fix},\, t=0}_{g=s}$$



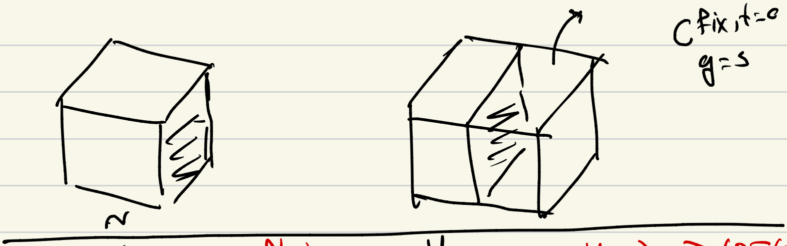
$$\mathbb{A}^1$$

$$\tilde{Y}_1 \quad Z(DT(\tilde{Y}_1)) = Y_1 \quad Z(DT(Y_{1/S})) \cdot Z(DT(\tilde{N}^{C}_{/S}))$$

$$Z(DT(\tilde{Y}_1)) = Z(DT(Y_{1/S}, \alpha)) \cdot Z(DT(\mathrm{Hilb}_n(\mathbb{P}^2/C)))$$

$\mathrm{Bl}\ N^{C} =: \tilde{N}^{C}$

$C^{\mathrm{fix},t=0}_{g=s}$



$\mathbb{A}^1$

$$\tilde{Y}_1 \quad Z(DT(\tilde{Y}_1)) = Y_1 \quad Z(DT(Y_{1/S})) \cdot Z(DT(\tilde{N}^{C}/S))$$

$$Z(DT(\tilde{Y}_1)) = Z(DT(Y_{1/S}, \alpha)) \cdot Z(DT(\mathrm{Hilb}_n(\mathbb{P}^2/C)))$$

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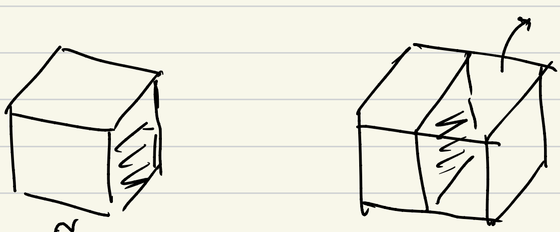
V.V modular form of weight $\left(\frac{-3}{2}\right)$

$\mathrm{Bl}\, N^C =: \tilde{N}^C$

$C^{fix,\, t=0}$

$g = s$



$\mathbb{A}^1$

$\tilde{Y}_1 \quad Z(DT(\tilde{Y}_1)) = Y_1 \quad Z(DT(Y_1/S)) \cdot Z(DT(\tilde{N}_C/S))$

$$Z(DT(\tilde{Y}_1)) = Z(DT(Y_1/S, \alpha)) \cdot Z(DT(\mathrm{Hilb}_n(\mathbb{P}^2/C)))$$

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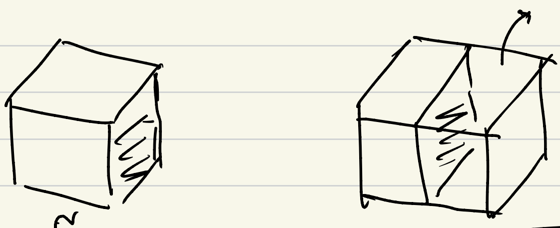
V.V modular form of weight $\left(\frac{-3}{2}\right)$

$$Z(DT(X)) = Z(DT(Y_1/S), \alpha) \cdot Z(DT(\mathrm{Hilb}_n(\mathrm{Bl}\,\mathbb{P}^2_{2pt}/C)))$$

$$= \frac{Z(DT(\tilde{Y}_1))}{Z(DT(\cancel{\mathrm{Hilb}_n}(\mathbb{P}^2/C)))} \cdot Z(DT(\cancel{\mathrm{Hilb}_n}(\mathrm{Bl}\,\mathbb{P}^2_{2pt}/C)))$$

$$= Z(DT(\tilde{Y}_1))$$ $\sim$ V.V modular form of weight $\left(\frac{-3}{2}\right)$ Q.e.d

$\mathrm{Bl}_{C^{\mathrm{fix},\,t=0}_{g=S}} N^C =: \tilde{N}^C$



$\mathbb{A}^1$

$$Y_1 \quad Z(DT(\tilde{Y}_1)) = Y_1 \quad Z(DT(Y_{1/S})) \cdot Z(DT(\tilde{N}^C/S))$$

$$Z(DT(\tilde{Y}_1)) = Z(DT(Y_{1/S}, \alpha)) \cdot Z(DT(\mathrm{Hilb}^n(\mathbb{P}^2/C)))$$

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V.V modular form of weight $\left(\frac{-3}{2}\right)$

$$Z(DT(X)) = Z(DT(Y_{1/S}), \alpha) \cdot Z(DT(\mathrm{Hilb}^n(\mathrm{Bl}\,\mathbb{P}^2_{2pt}/C)))$$

$$= \frac{Z(DT(\tilde{Y}_1))}{\cancel{Z(DT(\mathrm{Hilb}^n(\mathbb{P}^2/C)))}} \cdot \cancel{Z(DT(\mathrm{Hilb}^n(\mathrm{Bl}\,\mathbb{P}^2_{2pt}/C)))}$$

Thank You!

$$= Z(DT(\tilde{Y}_1)) \sim \text{V.V modular form of weight } \left(\frac{-3}{2}\right) \quad \text{Q.e.d}$$