tt^* Geometry, Heat Kernel Analysis and LG-type BCOV Invariants

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Outline

1. Analytic torison of compact Kähler manifolds
2. L^2 Hodge theory for Landau-Ginzburg model
3. Heat Analysis and LG type BCOV Invariants

Outline

Hirzebruch-Riemann-Roch Theorem

- (X, h): a compact Kähler manifold, $\dim_{\mathbb{C}} X = n$
- $E \to X$: a holomorphic vector bundle

We have the $\bar{\partial}_E$-complex

$$0 \to \mathcal{A}^{0,0}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{0,1}(E) \to \cdots \to \mathcal{A}^{0,n-1}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{0,n}(E) \to 0.$$

Theorem (Hirzebruch-Riemann-Roch)

$$\chi(E) = \sum_{q=0}^{n} (-1)^q \dim H^{0,q}(E) = \int_M \mathrm{Td}(TX)\,\mathrm{ch}(E).$$

Zeta function and regularized determinant

Equip with E a Hermitian metric, then for each q,

$$\Delta_E^q = \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E : \mathcal{A}^{0,q}(E) \to \mathcal{A}^{0,q}(E).$$

It has purely discrete spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$.

- $\zeta_q(s) = \sum_{\lambda_j > 0} \lambda_j^{-s}$:
  - ◇ well-defined for $\mathrm{Re}(s) \gg 0$,
  - ◇ meromorphic extension which is regular at $s = 0$. Formally,

$$\zeta_q'(0) \text{ “ = ” } - \sum_{\lambda_j > 0} \log \lambda_j.$$

- $\det{}'(\Delta_E^q) = \exp\left(-\zeta_q'(0)\right)$: regularized determinant.

Ray-Singer torsion

Definiton (Ray-Singer torsion, or $\bar{\partial}$-torsion)

$$T(E) := \prod_{q=0}^{n} \left(\det{}'(\Delta_E^q)\right)^{(-1)^q q} := \exp\left(-\sum_{q=0}^{n} (-1)^q \zeta_q'(0) \right).$$

Via the Mellin transform, we also write zeta function as

$$\zeta_q(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \operatorname{Tr} q \left(e^{-t\Delta_E^q} - \Pi \right) dt.$$

Let N be the number operator $N\big|_{\mathcal{A}^{0,q}(E)} = q\,\mathrm{id}$,

$$\begin{aligned} T(E) &\approx \exp\left(-\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \operatorname{Tr}(-1)^N N (e^{-t\Delta_E} - \Pi) dt \right) \\ &\approx \exp\left(-\int_0^{\infty} \operatorname{Tr}\left[(-1)^N N (e^{-t\Delta_E} - \Pi) \right] \frac{dt}{t} \right). \end{aligned}$$

Torsion for $\mathbb{C}/\Gamma$

Given the pair (W, χ):

- $W = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$,
- χ is a non-trivial character:

$$\chi(m\tau + n) = \exp[2\pi i(mu + nv)], \quad 0 < u, v < 1.$$

Then

$$T(W, \chi) = \left| e^{\pi i v^2 \tau} \frac{\theta_1(u - \tau v)}{\eta(\tau)} \right|.$$

The computation starts from the eigenvalues of Δ on $\mathcal{A}^0(L(\chi))$, which are given by

$$\lambda_{m,n} = -\frac{4\pi^2}{(\mathrm{Im}\tau)^2} |u + m - \tau(v + n)|^2.$$

Family case

Consider

- a holomorphic family $\pi : \mathcal{X} \to B$ of Kähler manifold
- a holomorphic vector bundle $\mathcal{E} \to \mathcal{X}$

For each $b \in B$, we have the restriction

$$E_b = \mathcal{E}\big|_{\mathcal{X}_b} \to \mathcal{X}_b$$

and the cohomology $H^*(\mathcal{X}_b; E_b)$.

Theorem (Grothendieck-Riemann-Roch)

$$\mathrm{ch}(R\pi_*\mathcal{E}) = \int_{\mathcal{X}/B} \mathrm{Td}(T\mathcal{X}/B)\,\mathrm{ch}(\mathcal{E}).$$

Torsion form and BGS-, BK-anomaly formula

By identifying the cohomology classes with harmonic forms,

$$\mathrm{ch}(R\pi_*\mathcal{E}) - \int_{\mathcal{X}/B} \mathrm{Td}(T\mathcal{X}/B)\,\mathrm{ch}(\mathcal{E})$$

becomes an exact differential form.

Theorem (Bismut-Gillet-Soulé, Bismut-Kohler)

There exists a differential form $T(\mathcal{X};\mathcal{E}) \in \mathcal{A}^*(B)$ satisfying

- $\frac{\partial_B\bar{\partial}_B}{2\pi i} T(\mathcal{X},\mathcal{E}) = \mathrm{ch}(R\pi_*\mathcal{E}) - \int_{\mathcal{X}/B} \mathrm{Td}(T\mathcal{X}/B)\,\mathrm{ch}(\mathcal{E})$;
- $T(E_b) = \exp(-T^0(\mathcal{X},\mathcal{E})\big|_b)$.

BCOV torsion

Consider the special case $E = E_p = \wedge^p T^* M$ and the corresponding $T_p := T(E_p)$, BCOV proposed the following combination of analytic torsion:

$$T_{\mathrm{BCOV}} := \prod_{p=0}^{n} T_p^{(-1)^p p} = \prod_{0 \le p,q \le n} \left(\det{}'(\Delta_{p,q})\right)^{(-1)^{p+q} pq}$$
$$\approx \exp\left(-\int_0^{+\infty} \mathrm{Tr}\left[(-1)^{N_1+N_2} N_1 N_2 (e^{-t\Delta} - \Pi)\right] \frac{dt}{t}\right),$$

where Δ is the Hodge laplacian, $N_1\big|_{\mathcal{A}^{p,q}} = p\,\mathrm{id}$, $N_2\big|_{\mathcal{A}^{p,q}} = q\,\mathrm{id}$.

Holomorphic anomaly equation

Theorem (BCOV, Fang-Lu-Yoshikawa)

Consider a family of Calabi-Yau manifolds, there is

$$\frac{\partial_B \bar{\partial}_B}{2\pi i} \log T_{\mathrm{BCOV}} = \operatorname{tr} C\bar{C} - \frac{\chi(X)}{12}\omega_{\mathrm{WP}},$$

here $\operatorname{tr} C\bar{C}$ *is the (generalized) Hodge metric and* ω_{WP} *is the Weil-Petersson metric.*

It can be proved by using BGS theorem for E_p and the fact that the fibers are Calabi-Yau.

Applications

- BCOV: From the point of view of topological string theory, $\log T_{\mathrm{BCOV}}$ is almost the B-model $g = 1$ partition function.

$$\log T_{\mathrm{BCOV}} \approx F_1.$$

  The HAE provides methods to calculate F_1. Via the mirror symmetry, it corresponds to the $g = 1$ GW invariants.

- FL: They used this relation to
  - ◇ prove certain kind of moduli spaces of polarized Calabi-Yau manifolds do not admit complete subvarieties,
  - ◇ give an estimate of $\partial\bar{\partial}\log T_{\mathrm{BCOV}}$ and prove that $\mathrm{tr}\, C\bar{C}$ is bounded by the Poincaré metric.

Outline

Landau-Ginzburg model

From the principle of LG/CY correspondence, we aim to discuss similar objects and structures in the LG side.

- L^2 Hodge theory of LG model:
  - ◇ Hodge decomposition,
  - ◇ tt^* geometry
- LG type torsion invariant and anomaly equation

A Landau-Ginzburg (LG) model is the theory of a pair (X, f), where

- X is a complex manifold;
- f is a holomorphic function on X, called the superpotential.

Bounded geometry and tame condition

Usually, we add more conditions for (X, h, f):

- (X, h) is a non-compact complete Kähler manifold with bounded geometry;
- f satisfies some "tame" condition to control the critical points of f and its growth.
  - ◇ Tame [Broughton]: there exists a compact set K, such that

$$|\nabla f| > 0, \text{ outside } K.$$

  - ◇ weakly tame [Némethi-Sabbah]: cohomologically tame or M-tame: "no modification of topology of fibers of f comes from infinity."

Tame conditions associated to $|\nabla^2 f|$

More requirements involving $\nabla^2 f$, such as

- tame condition of ellipticity [Klimek-Lesniewski, ($\mathbb{C}^n, f$ polynomial)]:
  1. $|\partial f| \to \infty$, as $z \to \infty$;
  2. For any $\epsilon > 0$, there is a constant C such that

$$|\partial^2 f| \le \epsilon |\partial f|^2 + C, \quad \forall z \in \mathbb{C}.$$

- strongly tame condition [Fan]: for any $C > 0$,

$$|\nabla f|^2(p) - C|\nabla^2 f|(p) \to \infty, \quad d(p, p_0) \to \infty$$

- Tame condition [Dai-Yan]:

$$\lim_{p\to\infty} |\nabla f| = \infty, \quad \lim_{p\to\infty} \frac{|\nabla^2 f|}{|\nabla f|^2 + 1} = 0.$$

Strongly elliptic condition

- strongly elliptic condition [Li-Wen]: For any $\epsilon > 0$, $k \geq 2$,

$$\epsilon|\nabla f|^k - |\nabla^k f| \to +\infty, \quad z \to \infty.$$

With this stronger condition, they found a nice space $\mathrm{PV}_{f,\infty}(X)$, which is closed under $\bar{\partial}_f$, wedge product and decomposition into components of Hodge degrees.

Important examples

- $X = \mathbb{C}^n$, $h = \frac{1}{2}\sum_{\nu=1}^{n}(dz_\nu \otimes d\bar{z}_\nu + d\bar{z}_\nu \otimes dz_\nu)$ and f is a non-degenerate quasi-homogenous polynomial on $\mathbb{C}^n$.
- $X = (\mathbb{C}^*)^n$, $h = \frac{1}{2}\sum_k \left(\frac{dz_k}{z_k} \otimes \frac{d\bar{z}_k}{\bar{z}_k} + \frac{d\bar{z}_k}{\bar{z}_k} \otimes \frac{dz_k}{z_k}\right)$ and f is a non-degenerate convenient Laurent polynomial on $(\mathbb{C}^*)^n$.
- $X = \mathbb{C}^n/G$, and f is a G-invariant polynomial on $\mathbb{C}^n$ with an isolated singularity at the origin.

[Li-Wen]: The strongly elliptic condition holds for these three cases.

Saito's theory

The mathematical origin of LG B-model is due to K. Saito's theory (1982) of primitive form over the universal unfolding of an isolated hypersurface singularity.

- Brieskorn lattice

$$\hat{H}_f = \Omega_X^n[[s]]/(sd + df\wedge)\Omega_X^{n-1}[[s]]$$

- Higher residue pairing

$$K_f : \hat{H}_f \times \hat{H}_f \to s^n\mathbb{C}[[s]].$$

- primitive form ξ (a family of holomorphic volume form)

$$\left(s\frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial t^\beta} - C_{\alpha\beta}^{\gamma}\frac{\partial}{\partial t^\gamma}\right)\int_\Gamma e^{F/s}\xi = 0.$$

Barannikov-Kontsevich's theory

Geometrically, Saito's theory is the variation of $\frac{\infty}{2}$-Hodge structures, a notion due to Barannikov-Kontsevich's approach to deformation of compact Calabi-Yau. They considered the dGBV

$$\left(\mathrm{PV}(X), \Omega_X, \bar{\partial}, \partial, [,]\right)$$

where

- $\mathrm{PV}(X) = \Omega^{0,*}(X, \wedge^* TX)$;
- ∂ is the BV operator w.r.t. the holomorphic volume form Ω_X;
- $[,]$ is the Schouten-Nijenhuis bracket.

Bogomolov-Tian-Todorov lemma + pairing structure + Hodge-to-de Rham degeneration lead to a smooth moduli of deformations that carries a FM structure.

Li-Wen's work

In the Landau-Ginzburg model, X is non-compact. Li-Wen found a nice space to study the dGBV algebra

$$\left(\mathrm{PV}_{f,\infty}(X), \bar{\partial}_f, \partial, [\,,\,]\right)$$

where $\mathrm{PV}_{f,\infty}(X)$ enjoys both nice integration (L^2 pairing) and Hodge decompositions. Then they can apply BK construction on compact CY to LG directly.

Calabi-Yau type LG model

Let $X = \mathbb{C}^n, f = z_1^n + \cdots + z_n^n$. If we view $[z_1, \ldots, z_n]$ as the homogeneous coordinates of $\mathbb{P}^{n-1}$, then $f = 0$ defines a Calabi-Yau hypersurface X_f in $\mathbb{P}^{n-1}$ and the central charge of f is

$$\hat{c}_f = n - 2 = \dim_{\mathbb{C}} X_f.$$

Consider the deformation for f of the form

$$F = f + uz_1 \cdots z_n.$$

Such a deformation has a global $\mathbb{C}^*$ action, and hence induces a complex deformation of the hypersurface.

From the point of view of complex structure of X_f:

- $\bar{\partial}$, Kähler identity, Δ
- Hodge decomposition and Hodge filtration
- Deformation of complex structure (VHS, VMHS)
  - Gauss-Manin connection
  - Weil-Petersson metric, tt^* metric, Hodge metric
  - period integral
- Heat kernel theory of $e^{-t\Delta}$
  - local analysis, index theorem
  - analytic torsion (Ray-Singer torsion, BCOV torsion), Cheeger-Müller theorem, Bismut-Zhang theorem, Quillen anomaly equation

Question

How to read the information of X_f from f?

1d LG model

In the Schrödinger representation of 1d LG model (SQM):

- The Hilbert space is given by $L^2\mathcal{A}(X)$, i.e. the space of L^2-integrable forms on X. The inner product is given by

$$g(\alpha,\beta) = \int_X \alpha \wedge *\bar{\beta}, \quad \alpha,\beta \in L^2\mathcal{A}(X).$$

  Here $*$ is the Hodge star operator w.r.t. the metric h.

- The charge operators Q_+, Q_-, Q_+^*, Q_-^* are represented by

$$\begin{aligned} Q_+ &= \bar{\partial}_f := \bar{\partial} + df\wedge, & Q_- &= \partial_f := \partial + d\bar{f}\wedge, \\ Q_+^* &= \bar{\partial}_f^* := - * \partial_{-f} *, & Q_-^* &= \partial_f^* := - * \bar{\partial}_{-f} * . \end{aligned}$$

- The Hamiltonian operator is given by

$$\Delta_f = \bar{\partial}_f \bar{\partial}_f^* + \bar{\partial}_f^* \bar{\partial}_f = \partial_f \partial_f^* + \partial_f^* \partial_f.$$

- The Fermion operator N: $N\big|_{\mathcal{A}^k(X)} = k\,\mathrm{id}.$

These operators

$$\left(\bar{\partial}_f, \bar{\partial}_f^*, \partial_f, \partial_f^*, \Delta_f, N\right)$$

satisfy the supersymmetric algebraic relations.

$$\begin{aligned}
&\partial_f^2 = \partial_f^{*2} = \overline{\partial}_f^2 = \overline{\partial}_f^{*2} = 0, \\
&\{\overline{\partial}_f, \partial_f\} = \{\overline{\partial}_f^*, \partial_f^*\} = \{\partial_f, \overline{\partial}_f^*\} = \{\overline{\partial}_f, \partial_f^*\} = 0, \\
&\{\partial_f, \partial_f^*\} = \Delta_f, \quad \{\overline{\partial}_f, \overline{\partial}_f^*\} = \Delta_f, \\
&[N, \overline{\partial}_f] = \overline{\partial}_f, \quad [N, \partial_f] = \partial_f, \\
&[N, \overline{\partial}_f^*] = -\overline{\partial}_f^*, \quad [N, \partial_f^*] = -\partial_f^*, \quad [N, \Delta_f] = 0.
\end{aligned}$$

In particular, Δ_f is a semi-positive definite self-adjoint real operator with respect to the pairing g.

Some observations on Δ_f

- Δ_f has the local expression

$$\Delta_f = \Delta + (h^{\bar{\mu}\nu}\nabla_\nu f_l \iota_{\partial_{\bar{\mu}}} dz^l \wedge + \overline{h^{\bar{\mu}\nu}\nabla_\nu f_l \iota_{\partial_{\bar{\mu}}} dz^l \wedge}) + |\nabla f|^2.$$

For the simplest case $\left(\mathbb{C},\ h,\ f = \frac{z^2}{2}\right)$, Δ_f is the complex 1 dimensional harmonic oscillator.

- special relation: $*\Delta_f = \Delta_{-f}*$ $\quad (*\Delta_f^k = \Delta_{-f}^{2n-k}*)$.
- The relation with Witten deformation:

$$d_f := e^{-f} \cdot d \cdot e^f = d + df\wedge, \quad \Box_f = d_f d_f^* + d_f^* d_f.$$

By Kähler identity,

$$\Delta_f = \frac{1}{2}\Box_{2\,\mathrm{Re}(f)}.$$

Analysis method in LG model

In the following, I will talk about

- Spectrum of Δ_f;
- Hodge decomposition for the twisted operators;
- spectral flow and $U(1)$ charge
- Lefschetz thimble and period integral
- heat kernel theory of $e^{-t\Delta_f}$ and torsion theory

Spectrum of Δ_f

Theorem (Klimek-Lesniewski, Fan)

Suppose that (X, h, f) is a non-compact complete Kähler manifold with bounded geometry and satisfies elliptic tame or strongly tame condition. Then

- Δ_f *has purely discrete spectrum and*
- *all the eigenforms form a complete basis of the Hilbert space* $L^2\mathcal{A}(X)$.

Comparison theorem

There are natural relations for the following complexes:

- Complexes in the smooth sense:

$$(\mathcal{A}_c^*(X), \bar{\partial}_f) \hookrightarrow (\mathcal{A}_{(2)}^*(X), \bar{\partial}_f) \hookrightarrow (\mathcal{A}^*(X), \bar{\partial}_f),$$

  which induces

$$H^*_{c,\,\bar{\partial}_f}(X) \to H^*_{(2),\bar{\partial}_f}(X) \to H^*_{\bar{\partial}_f}(X).$$

- L^2 cohomology with strong closure $\overline{\bar{\partial}_f}$:

$$(\mathcal{A}_{(2)}^*(X), \bar{\partial}_f) \hookrightarrow (L^2\mathcal{A}^*(X), \overline{\bar{\partial}_f}),$$

  which induces (an isomorphism by elliptic regularity)

$$\iota_{(2)} : H^*_{(2),\bar{\partial}_f}(X) \to H^*_{(2),\sharp,\overline{\bar{\partial}_f}}(X).$$

Why L^2?

Using the L^2 cohomology theory, one can extend the success of Hodge theory for compact oriented Riemann manifolds to less well-behaved spaces

- Non-compact manifold, even incomplete Riemann manifold
  - Hodge map
  - Hodge decomposition, Kodaira decomposition
- Nonisolated conical singularities, e.g. metric cone
- Dual to intersection homology

Hodge type theorem

Theorem (Klimek-Lesniewski, Fan, Li-Wen)

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a non-degenerate quasi-homogeneous polynomial and $\mathcal{H}^* := \ker \Delta_f$ be the space of harmonic forms on X. Then

- $\dim \mathcal{H}^* < \infty$.
- *(Hodge decomposition) There are orthogonal decompositions*

$$L^2\mathcal{A}^*(X) = \mathcal{H}^* \oplus \mathrm{Im}(\bar\partial_f) \oplus \mathrm{Im}(\bar\partial_f^*).$$

More precisely, there is a self-adjoint compact operator G_f on $L^2\mathcal{A}^*(X)$ such that

$$L^2\mathcal{A}^k(X) = \mathcal{H}^k \oplus \bar\partial_f\left(\bar\partial_f^* G_f L^2\mathcal{A}^k(X)\right) \oplus \bar\partial_f^*\left(\bar\partial_f G_f L^2\mathcal{A}^k(X)\right).$$

Hodge type theorem, continued

- There is a canonical isomorphism

$$H^k_{(2),\,\bar{\partial}_f}(X) \cong H^k_{\bar{\partial}_f}(X) \overset{\text{[Li-Li-Saito]}}{\cong} H^k_{c,\,\bar{\partial}_f}(X), \quad 0 \le k \le 2n.$$

- (Vanishing Theorem [Klimek-Lesniewski])

$$\mathcal{H}^k \cong H^k_{(2),\bar{\partial}_f}(X) \cong \begin{cases} 0 & \text{if } k \neq n \\ \Omega^n(X)/df \wedge \Omega^{n-1}(X) \cong \mathrm{Jac}(f) & \text{if } k = n, \end{cases}$$

where $\mathrm{Jac}(f) = \mathbb{C}[z_1, \ldots, z_n]/\langle \partial_{z_1} f, \ldots, \partial_{z_n} f \rangle$.

Spectral flow

Let $\phi \in \mathrm{Jac}(f)$. Then $\phi dz_1 \cdots dz_n$ is $\bar{\partial}_f$-closed, we have "Hodge decomposition" (via a representative in $H_{c,\bar{\partial}_f}(X)$)

$$\phi dz_1 \cdots dz_n = \omega_\phi + \bar{\partial}_f \beta_\phi, \tag{2.3}$$

where ω_ϕ is Δ_f-harmonic and β_ϕ has at most polynomial growth.

Proposition

The map $S : \mathrm{Jac}(f) \to \mathcal{H}$ given by $\phi \mapsto \omega_\phi$ in (2.3) is a well-defined linear isomorphism.

Bi-grading on $\mathcal{H}$

Let $\alpha = \alpha(z, \bar{z}) dz_{i_1} \cdots dz_{i_p} d\bar{z}_{j_1} \cdots d\bar{z}_{j_q}$ be a (p, q)-form on C^n, for $\theta \in \mathbb{R}$, define

$$\mathcal{T}(\theta)\alpha = e^{i(-\sum_k Q_{i_k} + \sum_l Q_{j_l})\theta} \alpha(e^{-iQ_j\theta} z_j, e^{iQ_j\theta} \bar{z}_j) dz_{i_1} \cdots dz_{i_p} d\bar{z}_{j_1} \cdots d\bar{z}_{j_q},$$

where $Q_j = q_j d$, $q_j = \frac{a_j}{b_j}$ with $(a_j, b_j) = 1$, $d = (b_1, \ldots, b_n)$.
Furthermore, we define

$$\mathcal{P}(\theta)(\alpha) = e^{idp\theta} \mathcal{T}(\theta)\alpha, \quad \mathcal{Q}(\theta)(\alpha) = e^{idq\theta} \mathcal{T}(-\theta)\alpha.$$

Proposition (T-Yan)

The two operators $\mathcal{P}$ and $\mathcal{Q}$ are unitary operators. Furthermore,

1. $[\mathcal{P}(\theta), \bar{\partial}_f] = [\mathcal{P}(\theta), \bar{\partial}_f^\dagger] = [\mathcal{P}(\theta), \Delta_f] = 0;$
2. $[\mathcal{Q}(\theta), \partial_f] = [\mathcal{Q}(\theta), \partial_f^\dagger] = [\mathcal{Q}(\theta), \Delta_f] = 0.$

Thus they give a unitary action on $\mathcal{H}$.

Bi-grading on $\mathcal{H}$

For simplicity, assume that $q_i = \frac{1}{n}$. Then $d = n$, $Q_i = 1$.

We say a harmonic form $\omega \in \mathcal{H}$ has bi-grading $(\hat{p}, \hat{q})$ if

$$\mathcal{P}(\theta)(\omega) = e^{in\hat{p}\theta}\omega, \quad Q(\theta)(\omega) = e^{in\hat{q}\theta}\omega.$$

In particular, $\mathcal{H}$ has a natural real structure given by complex conjugation.

$$\kappa : \mathcal{H} \to \mathcal{H}, \quad \omega \mapsto \bar{\omega}.$$

It is easy to check if ω has the bi-grading $(\hat{p}, \hat{q})$, then $\bar{\omega}$ has the bi-grading $(\hat{q}, \hat{p})$.

period integrals

Let ϕ be a homogeneous polynomial that represents an element in $\mathrm{Jac}(f)$.
There are two natural n-forms

$$e^f \phi dz_1 \cdots dz_n, \qquad e^{f+\bar{f}} \omega_\phi.$$

Both of them represent two classes in $H^n(\mathbb{C}^n, \{\mathrm{Re}(f) \ll 0\}; \mathbb{C})$ respectively.

Given a Lefschetz thimble $\gamma^- \in H_n(\mathbb{C}^n, \{\mathrm{Re}(f) \ll 0\}; \mathbb{Z})$, we obtain two kinds of period integrals:

$$\int_{\gamma^-} e^f \phi dz_1 \cdots dz_n, \qquad \int_{\gamma^-} e^{f+\bar{f}} \omega_\phi \quad \text{(Cecotti-Vafa)}$$

Question

What's the relation between the two kinds of period integrals?

Transition matrix

Theorem (T-Yan)

Let ϕ_a be a homogeneous basis of $\mathrm{Jac}(f)$ such that $\deg(\phi_a)$ is increasing, and $\omega_a = S(\phi_a)$. Then there exists a matrix T such that

$$[e^f \phi_a dz_1 \cdots dz_n] = T_{ab}[e^{f+\bar{f}} \omega_b] \text{ in } H^n(\mathbb{C}^n, \{\mathrm{Re}(f) \ll 0\}; \mathbb{C}).$$

More explicitly, let $l_{ab} := \frac{l(A_a) - l(A_b)}{n}$,

$$T_{aa} = 1 \quad \text{for } 0 \le a \le \mu - 1;$$

$$T_{ab} = \begin{cases} 0, & \text{if } l_{ab} \notin \mathbb{Z}^+; \\ \frac{(-1)^{l_{ab}}}{l_{ab}!} \sum_{l_c = l_b} \int_{\mathbb{C}^n} \bar{f}^{l_{ab}} \phi_a dz_1 \cdots dz_n \wedge *\bar{\omega}_c g^{cb}, & \text{if } l_{ab} \in \mathbb{Z}^+. \end{cases}$$

Here (g^{ab}) is the inverse of (g_{ab}), $g_{ab} := g(\omega_a, \omega_b)$. In particular,

$$\int_{\gamma_k^-} e^{f+\bar{f}} \omega_0 = \int_{\gamma_k^-} e^f dz_1 \wedge \cdots \wedge dz_n.$$

Deformation

Definiton

Let $\{\phi_i\}_{i=1}^{\mu_f}$ *be a homogeneous basis of* $\mathrm{Jac}(f)$*, the deformation*

$$F = f + \sum_{i=1}^{s} u^i \phi_i.$$

is called

- *relevant, if* $\mathrm{wt}(\phi_i) < 1$*, for* $i = 1, \ldots, s$*;*
- *marginal, if* $\mathrm{wt}(\phi_i) = 1$*, for* $i = 1, \ldots, s$*;*
- *irrelevant, if* $\mathrm{wt}(\phi_i) > 1$*, for some* i*.*

tt^* geometry

For relevant and marginal deformations, we can work with the family constructions of above pictures. Via the natural projections, one can define 4 natural operators on the Hodge bundle.

- $D_i = \Pi \circ \partial_i$, $\bar{D}_{\bar{j}} = \Pi \circ \bar{\partial}_{\bar{j}}$, where $\partial_i = \frac{\partial}{\partial u_i}$;
- $C_i = \Pi \circ \phi_i$, $\bar{C}_{\bar{j}} = \Pi \circ \bar{\phi}_{\bar{j}}$.

Theorem (tt^* equations)

1. $[C_i, C_j] = [\bar{C}_{\bar{i}}, \bar{C}_{\bar{j}}] = 0;$
2. $[D_i, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{j}}, C_i] = 0;$
3. $[D_i, C_j] = [D_j, C_i], \quad [\bar{D}_{\bar{i}}, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{j}}, \bar{C}_{\bar{i}}];$
4. $[D_i, D_j] = [\bar{D}_{\bar{i}}, \bar{D}_{\bar{j}}] = 0; \quad [D_i, \bar{D}_{\bar{j}}] = -[C_i, \bar{C}_{\bar{j}}].$

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The rigorous definition of LG type torsion

The existence of heat kernel function for $e^{-t\Delta_f}$ (Schwarz form space)

↓ L^2 extension

The operator $e^{-t\Delta_f}$ is Hilbert-Schmidt $\rightsquigarrow$ trace-class.

↓ when $\mathrm{Re}(s)$ is large enough

Define the i-th zeta functions $\zeta_f^i(s)$

↓ Meromorphic extension for s

Define the i-th singularity torsion $T^i(f)$

Existence of heat kernel

- According to the spectrum analysis, there exists a kernel function of $e^{-t\Delta_f}$;
- Local analysis: parametrix construction and the reminder estimate $\rightsquigarrow$ heat kernel expansion as $t \to 0+$

Since the base space is non-compact and the potential part is unbounded, then the usual perturbation way to construct the parametrix

$$K_N(z, w; t) = \frac{1}{(2\pi t)^n} e^{-\frac{|z-w|^2}{2t}} \sum_{a=0}^{N} V_a(z, w) t^a$$

does not work. To make sure the convergence, we need

re-summation

Resummation

Method: Resummation

Fix a large $N \in \mathbb{N}$, the approximate solution for the heat equation

$$(\partial_t + \Delta_f) p_t(z, w) = 0$$

is defined to be

$$K_N(z, w; t) = \frac{1}{(2\pi t)^n} e^{-\frac{|z-w|^2}{2t}} e^{-tg(z,w)} \sum_{a=0}^{N} U_a(z, w) t^a,$$

where

$$\boxed{g = 2 \int_0^1 |\partial f|^2 (\tau(z - w) + w) d\tau.}$$

Existence of the heat kernel

Moreover, we get the recursion relation for U_a $(a \geq 3)$ which depends on U_{a-1}, U_{a-2}, U_{a-3}:

$$
\begin{aligned}
&U_a(z,w) \\
&= \frac{1}{r^a}\int_0^r s^{a-1}\Big\{2\partial_v\bar{\partial}_{\bar{v}}U_{a-1}(z,w) - B(z)U_{a-1}(z,w) \\
&\quad - 2[\partial_v g\bar{\partial}_{\bar{v}}U_{a-2}(z,w) + \bar{\partial}_{\bar{v}}g\partial_v U_{a-2}(z,w) + \partial_v\bar{\partial}_{\bar{v}}gU_{a-2}(z,w)] \\
&\quad + 2\partial_v g\bar{\partial}_{\bar{v}}gU_{a-3}(z,w)\Big\}\,ds,
\end{aligned}
$$

where

- $r = |z-w|$ and
- $B(z)$ is the matrix representing the operator L_f.

heat kernel expansion

Theorem (Fan-Fang (non-deformed case), T)

Given the pair (f, F), where

- f *is a non-degenerate quasi-homogeneous polynomial on* $\mathbb{C}^n$ *such that* $q_M - q_m < \frac{1}{3}$,
- $F = F(z;u) = f(z) + \sum_{i=1}^{s} u^i \phi_i$ *is one of the following deformations:*
    - ⋄ *the marginal deformation of* f;
    - ⋄ *the relevant deformation of* f *with one more weight condition* wt$(\phi_i) < 1 - 2(q_M - q_m)$.

Let Δ_F *act on the Schwartz form space* $\mathcal{S}\mathcal{A}_B^*(\mathbb{C}^n)$, *with the norm* $\|\cdot\|_0$. *Then there exists a unique heat kernel function* $p_t(z, w; u)$ *of* $e^{-t\Delta_F}$, *which smoothly depends on the deformation parameters* $u, \bar{u}$.

In particular, if f is homogeneous, the result holds for its relevant or marginal deformation.

L^2 extension and heat trace

Proposition (T)

Under the weight condition for (f, F),

1. *The kernel function $p_t(z, w; u)$ is of L^2 class. Therefore, $e^{-t\Delta_F}$ is Hilbert-Schmidt, and*

$$\operatorname{Tr} e^{-t\Delta_F} = \int_{\mathbb{C}^n} \operatorname{tr} p_t(z, z; u) dvol_z.$$

In particular, when F is homogeneous of degree $p+1$, then $\operatorname{Tr} e^{-t\Delta_F}$ is a Laurent series of $t^{\frac{1}{2p}}$ with leading degree $-\frac{(p+1)n}{p}$. Let us order the powers by $\alpha_1^k = -\frac{(p+1)n}{p} < \alpha_2^k < \cdots < \alpha_{i_0(k)}^k = 0 < \alpha_{i_0(k)+1}^k < \cdots$,

$$\operatorname{Tr}(e^{-t\Delta_F^k}) = \sum_{1 \le a \le i_0(k)} c_{a,k}(u, \bar{u}) t^{\alpha_a^k} + \sum_{b > i_0(k)} c_{b,k}(u, \bar{u}) t^{\alpha_b^k}.$$

2. *For multiple indices I, J, K, L, the operators $\partial_z^I \bar{\partial}_{\bar{z}}^J \phi^K \bar{\phi}^L e^{-t\Delta_F}$ are also Hilbert-Schmidt.*

Supertrace and Zeta functions

For $i \in \mathbb{N}$, Fan-Fang defined the i-th zeta function to be

$$\begin{aligned}\zeta_f^i(s) &= \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(-1)^N N^i (e^{-t\Delta_f} - \Pi) dt \\ &= \frac{1}{2\Gamma(s)} \sum_{k=0}^{2n} (-1)^k \int_0^\infty t^{s-1} \operatorname{Tr} k^i (e^{-t\Delta_f} - \Pi_k) dt,\end{aligned}$$

where Π is the projection to the harmonic forms.

$i = 0$: local index theorem

Theorem (local index theorem)

$$\mathrm{Tr}(-1)^N(e^{-t\Delta_f} - \Pi) = 0, \quad \textit{then} \quad \zeta_f^0(s) = 0.$$

Recall for $(\mathbb{C}^n, h, f)$, we have

$$\Delta_f = -2\sum \partial_\nu \bar{\partial}_{\bar{\nu}} + L_f + 2|\partial f|^2,$$

where

$$\boxed{L_f = -2(\partial_\mu \partial_\nu f \iota_{\bar{\mu}} dz^\nu \wedge + \overline{\partial_\mu \partial_\nu f \iota_{\bar{\mu}} dz^\nu \wedge}).}$$

$$\mathrm{str}\, L_f^m := \mathrm{tr}(-1)^N L_f^m = \begin{cases} 0 & 0 \le m < 2n \\ (2n)!(-1)^n 4^n |\,\mathrm{Hess}(f)|^2 & m = 2n. \end{cases}$$

$i = 1$: vanishing theorem

Theorem (Fan-Fang (non-deformed case), T)

Let (f, F) satisfy the above weight conditions,

$$\zeta_F^1(s) = 0.$$

Remark

- *The vanishing happens in the super-trace level.*
- *It is the analog of vanishing result for Ray-Singer torsion:*

$$\sum_p (-1)^p \sum_q \mathrm{Tr}(-1)^q q \left(e^{-t\Delta_{p,q}} - \Pi_{p,q} \right) = 0.$$

$i = 2$: Transgression formula

By using

- Duhamel formula + supersymmetric relations
- the acyclic property for Trace

Theorem (T, Transgression formula)

For $i = 2$, under the weight condition for (f, F), we have

$$\bar{\partial}_{\bar{j}}\partial_i \operatorname{Tr}(-1)^N N^2(e^{-t\Delta_F} - \Pi) = -2t\frac{d}{dt}\operatorname{Tr}(-1)^N \phi_i\bar{\phi}_{\bar{j}}e^{-t\Delta_F} + 2t\frac{d}{dt}\operatorname{Tr}(-1)^N \int_0^t e^{-t'\Delta_F}\bar{\phi}_{\bar{j}}e^{-(t-t')\Delta_F}\Delta_F\phi_i dt',$$

where the ∂_i and $\bar{\partial}_{\bar{j}}$ are taken with respect to the deformation parameters $\{u^i, \bar{u}^{\bar{j}}\}$.

Torsion type invariant

One can consider the regularization of $\zeta(s)$ for s, and define the i-th torsion invariant

$$\begin{aligned}\log T^i(f) =& -(\zeta_1^{R,i}+\zeta_2^i)'(0)\\ =& -\frac{1}{2}\int_0^1\left(\mathrm{Tr}(-1)^N N^i(e^{-t\Delta_f}-\Pi)-\sum_{1\le a\le i_0} d_{a;i}t^{\alpha_a}\right)\frac{dt}{t}\\ &-\frac{1}{2}\int_1^\infty \mathrm{Tr}(-1)^N N^i(e^{-t\Delta_f}-\Pi)\frac{dt}{t}\\ &-\frac{1}{2}\sum_{1\le a<i_0}\frac{d_{a;i}}{\alpha_a}+\frac{1}{2}\Gamma'(1)d_{i_0;i},\end{aligned}$$

where $\alpha_a,\alpha_b\in\mathbb{Q}$, $\alpha_1<\alpha_2<\cdots<\alpha_{i_0}=0<\alpha_{i_0+1}<\cdots$ and $d_{a;i}$ are given by

$$\mathrm{Tr}(-1)^N N^i(e^{-t\Delta_f}-\Pi)=\sum_{1\le a\le i_0} d_{a;i}t^{\alpha_a}+\sum_{b>i_0} d_{b;i}t^{\alpha_b},\quad t\to 0+.$$

Anomaly formula

For a function $P(t) = \sum_a P_a t^a$, set $(P(t))_a = P_a$.

Theorem (T, Holomorphic anomaly equation)

Let f be a non-degenerate homogeneous polynomial on $\mathbb{C}^n$ and F be its relevant or marginal deformation. Then

$$\bar{\partial}_{\bar{j}}\partial_i \log T^2(F) = (-1)^n \operatorname{tr} C_i \bar{C}_{\bar{j}} - \left(\operatorname{Tr}(-1)^N \int_0^t \phi_i e^{-t'\Delta_F} \bar{\phi}_{\bar{j}} e^{-(t-t')\Delta_F} dt' \right)_1,$$

where $C_i = \Pi \circ \phi_i$, $\bar{C}_{\bar{j}} = \Pi \circ \bar{\phi}_{\bar{j}}$. When F is the marginal deformation,

$$\begin{aligned} &\left(\operatorname{Tr}(-1)^N \int_0^t \phi_i e^{-t'\Delta_F} \bar{\phi}_{\bar{j}} e^{-(t-t')\Delta_F} dt' \right)_1 \\ &= -\frac{(-1)^n}{(2\pi)^n} \int_{\mathbb{C}^n} \phi_i \bar{\phi}_{\bar{j}} e^{-2|\partial F|^2} 4^n |\partial^2 F|^2 |\operatorname{Hess}(F)|^2. \end{aligned}$$

Remark. I

- (Dai-Yan): they study the Witten deformation of f on non-compact manifold (M, h) and Witten Laplacian

$$d_{Tf} = e^{-Tf} \circ d \circ e^{Tf}, \ T \geq 0, \qquad \Box_{Tf}.$$

  - If (M, h, f) satisfies certain tame conditions[Dai-Yan], and f is a Morse function, then the Witten instanton complex (for large enough T), Thom-Smale-Witten complex, and a certain relative de Rham complex are quasi-isomorphic.
  - The heat kernel expansion K_{Tf} for $\Box_{Tf}$.
  - Cheeger-Müller/Bismut-Zhang Type Theorem
  - gluing formula for the corresponding analytic torsion

Remark. II

- (Shen-Xu-Yu): they study the index theory and torsion form of LG model
  - $(-1)^n \mu_f = \mathrm{ind}(\bar{\partial}_f + \bar{\partial}_f^*) = \int_X \mathrm{Td}(X) \wedge \mathrm{ch}(df)$;
  - a pair of new grading operators $(\hat{N}_1, \hat{N}_2)$:

$$[\hat{N}_1, \bar{\partial}_f] = [\hat{N}_2, \partial_f] = 0,$$
$$[\hat{N}_1, \partial_f] = \partial_f, \quad [\hat{N}_2, \bar{\partial}_f] = \bar{\partial}_f.$$

  - torsion form $T(X, F) \in \mathcal{A}^*(B)$, with

$$T^0(X, F) \approx \int_0^\infty \mathrm{Tr}(-1)^N \hat{N}_1 \hat{N}_2 e^{-t\Delta_F} \frac{dt}{t}.$$

Remark. III

- Isomonodromic deformation:
  - ◇ tt^* geometry is explained as the isomonodromic deformation of a meromorphic connection on $\mathbb{P}^1$ with Poincaré rank 1 irregular singularities at $0, \infty$. (Cecotti-Vafa, Dubrovin, Hertling, Sabbah, Simpson,...)
  - ◇ Cecotti-Vafa explained the isomonodromic tau function as a new supersymmetric index

$$\log \tau \approx \int_{\mathcal{F}} \frac{d^2\rho}{\rho_2} \mathrm{Tr}[(-1)^F F_R F_L q^{H_L} \bar{q}^{H_R}].$$

  - ◇ In the Gromov-Witten theory, when the FM is semi-simple, Dubrovin-Zhang gave the relation between isomonodromic tau function and G-function (Getzler)

$$G = \log \tau_{FM} - \frac{1}{24} \log J.$$

Remark. IV

- extend the above analysis to the 2d supersymmetric quantum field theory.
  - ◇ Field theoretical analogs of $\bar{\partial}_f$ and ∂_f

$$Q_{+f} = \int_0^{2\pi} \bar{b}^*(\sigma)(i\bar{\pi}(\sigma) - \partial_\sigma\bar{\phi}(\sigma))d\sigma + \int_0^{2\pi} b^*(\sigma)\partial f(\phi(\sigma))d\sigma,$$
$$Q_{-f} = \int_0^{2\pi} b^*(\sigma)(i\pi(\sigma) + \partial_\sigma\phi(\sigma))d\sigma + \int_0^{2\pi} \bar{\phi}^*(\sigma)\overline{\partial f(\phi(\sigma))}d\sigma.$$

  - ◇ Infinite dimensional Kähler structure:

$$L = \int_0^{2\pi} b^*(\sigma)\bar{b}^*(\sigma), \quad \Lambda = \int_0^{2\pi} \bar{b}(\sigma)b(\sigma)d\sigma,$$
$$h = -\int_0^{2\pi} [b^*(\sigma)b(\sigma) - \bar{b}(\sigma)\bar{b}^*(\sigma)]d\sigma,$$
$$\rightsquigarrow Q^*_{+f}, Q^*_{-f}.$$

Thank You!