Irregular KZ equations from Liouville Conformal Blocks

(based on work w/ N. Reshetikhin, Y. Liu)

Motivation:

two-fold:

1) AGT correspondence
   (4d Argyres-Douglas theories give rise to irr. operators in 2d)

2) Topological phases of matter (TQFTs)
   $\rightarrow$ wave-functions of Anyon states are given by conformal blocks
   What if we include irr. "defects"?

Background:

We will consider 2d Liouville theory:

$$S = \int dt\, d\sigma \left(\frac{1}{16\pi} \left((\partial_t \varphi)^2 - (\partial_\sigma \sigma)^2 \right) - \mu e^{\varphi} \right)$$

$\rightarrow$ quantum theory ($\hbar = b^2$) flows to CFT with $b, \alpha \in \mathbb{C}$:

$$Q = b + \frac{1}{b}, \quad \hat{c} = 1 + 6Q^2, \quad \Delta_\alpha = \alpha(Q - \alpha)$$

vertex operators: $V_\alpha = e^{2\alpha \chi}$, $\chi \underset{b\to 0}{\sim} \frac{1}{2b}\varphi$

degenerate operators:

$$\Psi_{r,s} := \exp\left(\left[(1-r)b + \frac{1-s}{b}\right]\phi\right), \quad r,s \in \mathbb{N}^+$$

1) Irregular reps of Virasoro alg.:

In the simplest case of rank 1:

$$L_2 |I_{\alpha,c}\rangle = -\frac{c^2}{4}|I_{\alpha,c}\rangle$$
$$L_1 |I_{\alpha,c}\rangle = -c(\alpha - Q)|I_{\alpha,c}\rangle$$
$$L_0 |I_{\alpha,c}\rangle = (\Delta_\alpha + c\,\partial_c)|I_{\alpha,c}\rangle$$

α is Liouville momentum,
c arbitrary constant

$\rightarrow$ can show:

$: \exp(2\alpha\, \chi(L)) \exp(c\,\partial \chi(L)):$ with $L \to \infty$

is corresponding Vertex operator

Modified BPZ eq.:

consider now the degenerate primary

$$V_{-\frac{1}{2b}}(0)|0\rangle := |V_{-\frac{1}{2b}}\rangle \longrightarrow \Delta = -\frac{1}{2b}\left(Q + \frac{1}{2b}\right)$$

$\rightarrow$ null state: $(L_{-2} + b^2 L_{-1}^2)|V_{-\frac{1}{2b}}\rangle$

$$\mathcal{F}_{\Gamma}(z) := \langle V_{-\frac{K_0}{2b}}(0)\, V_{-\frac{1}{2b}}(z)\, V_{-\frac{K_1}{2b}}(1)\, I_{\alpha,\Lambda}(\infty)\rangle$$ conformal block

$\rightarrow$ modified BPZ eq.

$$\Big(b^2\frac{\partial^2}{\partial z^2} - \Big(\frac{1}{z}-\frac{1}{z-1}\Big)\frac{\partial}{\partial z} + \frac{\Delta(K_1)}{(z-1)^2} + \frac{\Delta(K_0)}{z^2}$$
$$+ \frac{1}{z(z-1)}\Big(c\frac{\partial}{\partial c} + \Delta_\alpha - \Delta(K_1) - \Delta(K_0) - \Delta(1)\Big)$$
$$- \frac{c(\alpha-Q)}{z} - \frac{c^2}{4}\Big)\mathcal{F}_{\Gamma} = 0$$

where $\alpha = mb + \frac{1+K_0+K_1}{2b}$ "charge conservation"

m: # screening charges

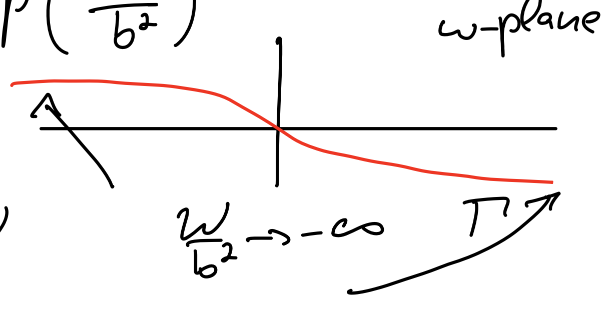
2) Integral representations

We introduce screening charges at positions w_i

$\rightarrow$ conformal blocks are given by integrals $(K_a \in \mathbb{Z})$

$$\mathcal{F}_{\Gamma_\sigma} = \int_{\Gamma_\sigma} W \underbrace{\Big(\prod_i V_{1/b}(w_i)\prod_a V_{-K_a/2b}(z_a)\, I_{\alpha,c}(\infty)\Big)}\prod_i dw_i$$

Wick contraction $=: \exp\left(\frac{W}{b^2}\right)$



Γ_σ: Lefschetz thimble

σ: critical point of W

$V_b(w_i)$ are excluded by charge conservation

we compute:

$$\mathcal{W}\left(\prod_i V_{1/b}(w_i)\prod_a V_{-K_a/2b}(z_a)\, I_{\alpha,c}(\infty)\right)$$

$$\sim \prod_{i\neq j}(w_i-w_j)^{-\frac{2}{b^2}}\prod_{i,a}(w_i-z_a)^{\frac{K_a}{b^2}}\prod_{a\neq b}(z_a-z_b)^{-\frac{K_aK_b}{2b^2}}$$

$$\times\exp\left(\frac{\Lambda}{b^2}\left(\sum_i w_i-\frac{1}{2}\left(\sum_a K_a z_a\right)\right)\right)$$

where we set $c=\frac{\Lambda}{b}$

3) Irregular KZ-equations

Let us define

$$\Phi_\Gamma(\vec{z}):=\int_\Gamma A(\vec{z},\vec{w})\,d^m w$$

where

$$A(\vec{z},\vec{w})=\exp\left[\frac{\Lambda}{b^2}\sum_i w_i-\frac{2}{b^2}\sum_{i<j}\log(w_i-w_j)+\sum_{i,j}\frac{K_j}{b^2}\log(w_i-z_j)\right]$$

$$\longrightarrow \mathcal{F}_\Gamma(\vec{z})=\Phi_0(\vec{z})\,\Phi_\Gamma(\vec{z})\quad\text{where}$$

$$\Phi_0(\vec{z})=\exp\left[-\frac{\Lambda}{2b^2}\sum_i K_i z_i-\sum_{i<j}\frac{K_iK_j}{2b^2}\log(z_i-z_j)\right]$$

Now for $\vec{m} = (m_1, m_2, \dots, m_N)$, with $m_i \in \mathbb{Z}_{\geq 0}$, $\sum_i m_i = m$, define

$$\phi^{(\vec{m})} = \int_{T_{\vec{m}}} A(\vec{z}, \vec{\omega}) \left(\prod_{i_1=1}^{m_1} \frac{1}{\omega_{i_1} - z_1} \cdots \prod_{i_N=1}^{m_N} \frac{1}{\omega_{i_N} - z_N} \right) d^m \omega$$

Then the following holds:

<u>Theorem 1:</u>

Let $\vec{\Psi}$ be a column vector containing all $\phi_0\, \phi^{(\vec{m})}$ with $\sum_i m_i = m$, then

$$b^2 \partial_{z_i} \vec{\Psi} = A_i \vec{\Psi} + \sum_{j \neq i} \frac{\Omega_{ij}^T}{z_j - z_i} \vec{\Psi},$$

where Ω^T is the transpose of the Ω-matrix for the height m $SL(2, \mathbb{C})$ KZ-equation and A_i acts as

$$A_i\, \phi^{(\vec{m})} = \frac{\Delta}{2} \left(2 m_i - K_i\right)$$

Idea of proof:

Let $v = v_1 \otimes v_2 \otimes \dots \otimes v_N$ be the highest weight vector for the tensor product of N $sl(2)$-representations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

with

$$\Omega_{ij} = \frac{H_i H_j}{2} + X_i Y_j + Y_i X_j$$

Define

$$F(\vec{z}, w) = \exp\left(\frac{\Lambda w}{b^2}\right) \prod_j (w - z_j)^{\frac{K_j}{b^2}} \sum_i \frac{Y_i}{w - z_i}$$

and

$$\Psi := \phi_0(z) \int \prod_{i \leq j} (w_i - w_j)^{-\frac{2}{b^2}} F(\vec{z}, w_1) \cdots F(\vec{z}, w_m) v \, d^m w$$

then one shows

$$b^2 \partial_i \Psi = \left(\frac{-\Lambda}{2} H_i + \sum_{j \neq i} \frac{\Omega_{ij}}{z_j - z_i} \right) \Psi$$

and $\phi_0 \phi^{(\vec{m})}$ is the coefficient of the basis vector

$$Y_1^{m_1} Y_2^{m_2} \cdots Y_N^{m_N} v$$

$\square$

One can use the irr. KZ to prove the irr. BPZ eq.

<u>Example:</u>

Let us consider the case of one screening charge and $K_0 = K_1 = 1$

$$\to \mathcal{F}_\Gamma = \Phi_0 \Phi_\Gamma$$

where

$$\Phi_\Gamma = \int_\Gamma dw\, e^{\frac{\Lambda w}{b^2}} \exp\left(\frac{\log(w-z) + \log(w-z_1) + \log(w-z_0)}{b^2}\right)$$

As Φ_0 satisfies the BPZ eq. by construction, one arrives at

$$b^2 \frac{\partial^2 \Phi}{\partial z^2} = \Lambda \frac{\partial \Phi}{\partial z} + \frac{1}{z-z_0}\left(\frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial z_0}\right) + \frac{1}{z-z_1}\left(\frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial z_1}\right)$$

But this is our KZ-equation!

$$\partial_i \Phi = K_i \Phi^{(1;i)}$$

and

$$\begin{pmatrix} K_i^{-1} & \\ & K_j^{-1} \end{pmatrix} \Omega_{ij} \begin{pmatrix} K_i & \\ & K_j \end{pmatrix} = \Omega_{ij}^T$$