Kähler submanifolds in Iwasawa manifold

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1. Complex nilmanifolds and Kähler manifolds

A **nilmanifold** is a compact manifold $N = G/\Lambda$, where G is a connected simply connected Lie group and $\Lambda \subset G$ is a cocompact lattice.
A **complex nilmanifold** is a pair (N, J) where N is a nilmanifold and J is an integrable G-invariant complex structure on N.

One of the reasons why complex nilmanifolds provoke such an interest in complex geometry is that most of them are very far from being Kähler:

- A complex nilmanifold admits a Kähler metric iff it is a torus.([**?**])
- If a complex nilmanifold is not a torus then its cohomology algebra is not formal.([**?**])
- There exist complex nilmanifolds with arbitrary long non-degenerating Frölicher spectral sequences.([**?**])

It is natural to ask how to describe Kähler submanifolds in complex nilmanifolds. We do it for a particular case of *Iwasawa manifolds*.

2. Iwasawa manifold

Consider the group $G = \{\left(\begin{smallmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{smallmatrix}\right) | z_i \in \mathbb{C}\}$ and a cocompact latiice $\Lambda \subset G$.
An **Iwasawa manifold** I is a quotient G/Λ. The standard complex structure on G gives a complex structure on I. Hence it is a compact complex homogenous three-dimensional manifold. Clearly it is a complex nilmanifold.
The group G is a central extension of a commutative 2-dimensional complex Lie group by 1-dimensional. The exact sequence

$$1 \to \mathbb{C} = Z(G) \to G \to G^{ab} = \mathbb{C}^2 \to 1$$

induces for any Iwasawa manifold I a holomorphic fibration $\pi \colon I \to T$ over some complex 2-dimensional torus (we call the **Iwasawa bundle**). It is a holomorphic principal E-bundle for the elliptic curve $E = Z(G)/(\Lambda \cap Z(G))$.

NB.: As it was noticed in a more general situation by Winkelmann ([**?**]), elliptic curve E always carries a complex multiplicaion and torus T is isogenic to a product of two copies of E.

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3. Kähler submanifolds in Iwasawa manifold

In [**?**] we prove the following theorems. Let I be an Iwasawa manifold, and let E and E' be two isogenic elliptic curves with complex multiplication (E is the fiber of the Iwasawa bundle).

The main theorems.

Theorem 1. Let $C \subset I$ be a smooth curve. Then there are two possibilities:

1.A) C is isomorphic to E;

1.B) C is contained in some holomorphic two-dimensional subtorus, isomorphic to $E \times E'$ for some elliptic curve E'.

Theorem 2. Let $S \subset I$ be a Kähler surface. Then there are two possibilities:

2.A) S is an abelian surface isomorphic to $E \times E'$ for some elliptic curve E';

2.B) S is a non-projective surface of algebraic dimension 1, diffeomorphic to $D \times E$ for a curve D of genus $g \geq 2$.

4.Holomorphic principal elliptic bundles

Let B be a complex manifold and $E = \mathbb{C}/\Gamma$ be an elliptic curve. Our aim is to study holomorphic principal E-bundles over B.
All of them are classified by the elements of $H^1(\mathcal{E}_B, B)$, where $\mathcal{E}_B$ is a sheaf of holomorphic functions with values in E.
For such a bundle $\mathcal{P}$ one might define (via the Chern-Weil theory) its **first Chern class** $c_1(\mathcal{P})$ with values in $H^2(B, \mathbb{C})$. These objects satisfy the following properties:

1. $H^1(B, \mathcal{E}_B)$ carries a natural structure of commutative group and $c_1 \colon H^1(B, \mathcal{E}_B) \to H^2(B, \mathbb{C})$ is a homomorphism.
2. If bundle $\mathcal{P}$ admits a multisection then its isomorphism class $[\mathcal{P}]$ lies in the torsion of $H^1(B, \mathcal{E}_B)$.
3. If $\underline{\Gamma} \subset \mathcal{O}_B$ is a subsheaf of functions with values in Γ, the exact triple
$$0 \to \underline{\Gamma} \to \mathcal{O}_B \to \mathcal{E}_B \to 0$$
induces the long exact sequence
$$\cdots \to H^1(B, \mathcal{O}_B) \to H^1(B, \mathcal{E}_B) \to H^2(B, \underline{\Gamma}) \to H^2(B, \mathcal{O}_B) \to \cdots$$
.
4. If B is Kähler then the total space of an E-bundle $\mathcal{P}$ is Kähler if and only if $c_1(\mathcal{P}) = 0$.([**?**])
5. For the Iwasawa bundle π (defined as in the Section 2) $c_1(\pi) \neq 0$. Moreover,
6. $c_1(\pi) \in H^{2,0}(T)$.

5.The main theorems: the sketch of the proof

The proof is based on the following lemma:

Lemma 3. *Assume that $C \subset I$ is a curve which projection $D = \pi(C) \subset T$ is also a curve. Then D is of genus one.*

The idea of the proof: Assume the opposite. The inclusion $i \colon D \hookrightarrow T$ factors through its Albanese variety $D \xrightarrow{alb} \operatorname{Alb} D \xrightarrow{\alpha} T$. The curve C provides the multisection of the restriction $i^*\pi$ of the Iwasawa bundle onto D. Passing to some fiberwise covering of π we can assume that it admits a section and hence trivial.
The Albanese morphism induces an isomorphism between the groups $H^1(D, \mathcal{E}_D)$ and $H^1(\operatorname{Alb}(D), \mathcal{E}_{\operatorname{Alb}(D)})$ (one can see it, for example, from the Property **??** in the Section 4). In particular, this means that the pullback of π to the Albanese variety is also trivial. But α is a surjective morphism of tori, so it is injective on the cohomology and

$$c_1(\alpha^*\pi) = \alpha^*(c_1(\pi)) \neq 0.$$

□

It is not hard to deduce the main theorems from this lemma. First of all notice that for any elliptic curve $E' \subset T$ the restriction $\pi|_{E'}$ is trivial. Then for any curve $C \subset I$ either $\pi(C)$ is a point (the case 1.A)) or $\pi(C)$ is an elliptic curve (the case 1.B)).
For any surface $S \subset I$ it is clear that $\pi(S) \subset T$ is a curve. Indeed, for the dimension reasons it is either a curve or the whole T, but π admits no multisection (properties **??**, **??**). Due to (prop. **??**) for any curve $D = \pi(S)$ the restriction of π on it has vanishing c_1. Hence (prop. **??**) its full preimage is a Kähler surface.

6.Conclusion

In [**?**] it was proved that any morphism from a complex nilmanifold to a porjective variety factors through a torus. The dual statement for Iwasawa manifold follows from our results:
If X is projective, then any holomorphic map $f \colon X \to I$ factors through a torus.
Conjecturally, this is true for any complex nilmanifold.

Yet, it is not clear what are the Kähler submanifolds in an arbitrary nilmanifold. If we replace G with the group of complex unipotent $n \times n$ matrices, most of our considerations stay correct but the computations become harder.

It is remarkable that the answer for the Iwasawa case is strongly connected with CM elliptic curves. What are the automorphisms of the Kähler submanifolds in nilmanifolds? What are their number-theoretical properties?