# Rectangular superpolynomials for the figure-eight knot 

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#### Abstract

We rewrite the recently proposed differential expansion formula for HOMFLY polynomials of the knot $4_{1}$ in arbitrary rectangular representation $R=\left[r^{s}\right]$ as a sum over all Young sub-diagrams $\lambda$ of $R$ with extraordinary simple coefficients $D_{\lambda^{t r}}(r) \cdot D_{\lambda}(s)$ in front of the $Z$-factors. Somewhat miraculously, these coefficients are made from quantum dimensions of symmetric representations of the groups $S L(r)$ and $S L(s)$ and restrict summation to diagrams with no more than $s$ rows and $r$ columns. They possess a natural $\beta$-deformation to Macdonald dimensions and produces positive polynomials, which can be considered as plausible candidates for the role of the rectangular superpolynomials. Both polynomiality and positivity are non-evident properties of arising expressions, still they are true. This extends the previous suggestions for symmetric and antisymmetric representations (when $s=1$ or $r=1$ respectively) to arbitrary rectangular representations. As usual for differential expansion, there are additional gradings. In the only example, available for comparison - that of the trefoil knot $3_{1}$, to which our results for $4_{1}$ are straightforwardly extended, - one of them reproduces the "fourth grading" for hyperpolynomials. Factorization properties are nicely preserved even in the 5-graded case.


## 1 Introduction

Superpolynomials are among the main mysteries of modern theoretical physics. There is a lot of evidence, that they exist, but there is neither a clear conceptual definition nor a clear algorithm for practical evaluation despite many efforts during the last two decades. This paper reviews the problem from the most naive direction and reports some small progress, based on the recent development in adjacent fields.

The story begins from a mysterious discovery that Wilson loop averages

$$
\begin{equation*}
\mathcal{H}_{R}^{\mathcal{K}}(A, q)=\left\langle\operatorname{Tr}_{R} P \exp \oint_{\mathcal{K}} \mathcal{A}\right\rangle \tag{1}
\end{equation*}
$$

in $3 d$ Chern-Simons theory [1], which depend only on the topology of the embedding $\mathcal{K} \hookrightarrow R^{3}$, i.e. on the topology of the knot $\mathcal{K}$, are Laurent polynomials with integer coefficients - when expressed in terms of peculiar variables $q=\exp \left(\frac{2 \pi i}{k+N}\right)$ and $A=q^{N}$ instead of the coupling constant $k$ and rank of the gauge group $S U(N)$. This fact explains the name knot polynomials [2] for $\mathcal{H}_{R}$, its various reductions at special values of $A$ and $q$ and generalizations to other gauge groups. $H_{R}$ per se is also known as HOMFLY polynomial, colored by a Young diagram $R$, which labels irreducible finite-dimensional representations of $S U(N)$. Integrality property implies that the coefficients of knot polynomials count something - these can be numbers of certain physical states in some stringy models [3], underlying the Chern-Simons theory, while in an abstract categorification program something means just dimensions of some vector spaces. Though $K$-theory allows "dimensions" to be negative, it would be simpler to have them positive - and this was the reason to look for generalization (refinement) of $H_{R}$, which should have all the coefficients positive. Convention is that this superpolynomial $\mathcal{P}_{R}^{\mathcal{K}}(A, q, T)$ has all coefficients positive integers and satisfies the reduction property

$$
\begin{equation*}
\mathcal{P}_{R}^{\mathcal{K}}(A, q, T=-1)=\mathcal{H}_{R}^{\mathcal{K}}(A, q) \tag{2}
\end{equation*}
$$

Knot-dependent complexes for which $\mathcal{P}$ and $H$ are respectively the Poincare and Euler polynomials are explicitly constructed for the single-box diagram $R=[1]=\square$ at fixed $N[4,5]$, but calculations are technically involved for
$N>2$ and there are only restricted successes for other representations $R$. An intriguing possibility is to consider superpolynomial as a $p$-adic generalization of HOMFLY [6]. In fact the number of additional arguments can be larger than two, $q \longrightarrow\{q,-q T\} \longrightarrow\left\{q_{i}\right\}$ are often thought as relatives to parameters of Kerov polynomials, and at least three-parametric deformations are already considered, both in the context of knot polynomials $[7,8]$ and for the closely related DIM-algebras [9, 10]. The literature on superpolynomials and their physical interpretation is rich, see, for example, [11]-[29].

If there is any commonly accepted definition today, then it describes the superpolynomial as

1) a Laurent polynomial $\mathcal{P}_{R}^{\mathcal{K}}(\mathbf{a}, \mathbf{q}, \mathbf{t})$ with all coefficients positive integers,
2) reproducing HOMFLY at $\mathbf{t}=-1$,
3) reproducing Khovanov-Jones polynomial at $\mathbf{a}=\mathbf{q}^{2}$
4) reproducing Khovanov-Rozansky at $\mathbf{a}=\mathbf{q}^{N}$

The problem with this "definition" is that Khovanov's is an explicit, but ad hoc construction, which is largely technical and not quite related with the "best" available conceptual definitions of HOMFLY - either from functional integrals and quantum-group theory or from various recursions. While HOMFLY are defined (at least mnemonically) in (1) as $3 d$ topological invariants, Khovanov's construction is in terms of knot diagrams - the knot's projections on $2 d$ planes, with topological invariance substituted by Reidemeister invariance, which is natural for representations of braids, but is only respected, but not fully explicit, in Khovanov's approach. Khovanov's complex is built with the help of a hypercube of colorings [4, 37, 38], and direct relation to the quantum- $\mathcal{R}$-matrix Reshetikhin-Turaev formalism [39, 40, 41] is available only at $N=2$, where a peculiar Kauffman's $\mathcal{R}$-matrix [42] can be used. Khovanov-Rozansky generalization to $N>2$ involves sophisticated matrix-factorization technique, which is very hard to use in practical calculations. Partly for this reason the construction of colored superpolynomials is still unclear, especially in non-rectangular representations.

Thus it is not a big surprise that there is a continuous search for a simpler and more straightforward definitions of superpolynomials and/or reformulations of HOMFLY calculus, which would allow to describe superpolynomials as natural and unambiguous deformations.

At advanced level the story is about refined Chern-Simons and, more generally, refined topological strings which is now linked to a deformation from loop Kac-Moody to toroidal Ding-Iohara-Miki algebras [10]. However, applications to knot theory are still work in progress. The real issue here is that knot theory in appropriate formulation is almost identical to representation theory, but is in fact a little more stable: knot theory can survive deformations which break the representation theory structures. The two recent examples are:
(i) applicability of Vogel's universality [30] to knots (knot invariants in the $E_{8}$-sector contain Vieta-like combinations of the roots of certain universal polynomials, rather than the roots themselves, which represent particular quantum dimensions - and thus possess simple universal expressions, even when dimensions fail to do so [31]), and
(ii) pretzel knot polynomials are expressed through bilinear combinations of Racah matrices ( $6 j$-symbols) [32], and these possess nice $\beta$-deformations [26], while this is not true for the $6 j$-symbols themselves.

Precise identification of this "healthy" part of representation theory, which in practice is captured by consideration of knot polynomials, remains a puzzling open question - and this is a probable reason for persisting problems with adequate definition and evaluation of super- and hyper-polynomials.

At naive level attempts are made to deform HOMFLY into superpolynomials for particular families of knots. If there is any canonical construction of this type, applicable universally to all knots, it should reveal and take into account the representation dependence of HOMFLY, because in any particular representation superpolynomials are believed to involve more knot invariants than HOMFLY, and balance is restored (if at all) only for the entire set of polynomials in all representations.

In this paper we report new results in the most promising approach of this kind - the one based on the differential expansions $[11,17,33,8,34,35]$. The point is that colored HOMFLY can be re-expanded in powers of the differentials $D_{n}=\left\{A q^{n}\right\}=A q^{n}-\frac{1}{A q^{n}}$, and these expansions have more pronounced representation dependence than original polynomials. Moreover, the differentials $D_{n}$ are directly related to differentials in Khovanov-Rozansky complexes [11], and the formalism naturally simplifies transition to superpolynomials. The method is most adequate for a family of twisted knots (see below), where it allowed to fully describe HOMFLY and superpolynomials in arbitrary symmetric representation and derive various recursions in $R$, which serve as a model for more complicated knots and links. Complexity of the differential expansion is regulated by a defect [35], which is actually a degree in $q^{2}$ of the fundamental Alexander polynomial in topological framing minus one. What distinguishes twisted knots is that they have defect zero and the differential expansion is actually
strengthened to one in $Z$-factors, which are bilinear combinations $Z_{n \mid m}=D_{n} D_{-m}=\left\{A q^{n}\right\}\left\{A / q^{m}\right\}$, and have a very clear superpolynomial deformation to $\mathcal{Z}_{n \mid m}=\left\{A q^{n}\right\}\left\{A / t^{m}\right\}$.

In this paper we use the recently discovered HOMFLY polynomials in all rectangular representations and their differential expansion for the figure-eight knot $4_{1}[36]$ to conjecture the corresponding rectangular twisted superpolynomials, which are in accord with various previous speculations. Also straightforward in this case is the next, forth grading [7], which seems to be naturally built into the differential-expansion formalism [8].

Comparison with alternative approaches, involving $c$-factors [13], DAHA algebra (a part of DIM) [14, 25, 24], Verlinde algebras [12], $T$-deformed $\mathcal{R}$-matrices [43, 44] etc is very desirable, but such strong results are not yet available there. Hopefully this paper will stimulate new progress in these other directions. Also of interest are generalizations to other defect-zero knots. Really challenging remain the cases of non-vanishing defects and especially of non-rectangular representations, where the structure of differential expansion remain obscure and sometime questioned is the very existence of superpolynomials (see [47, 48, 49] for partial description of the simplest $R=[21]$ and $R=[31]$ cases). Last, but not the least, reduction of superpolynomials to finite $N$ and the difference between reduced and unreduced superpolynomials are not yet captured by differential expansion tools - therefore in this paper we deal only with reduced superpolynomials at generic $A$.

The main results of this paper are eqs.(17) and (32).
The first is a strong and inspiring reformulation of the recent result of [36], and it is based on an amusing decomposition formula (9) for binomial coefficients, specific for rectangular representations.

The second is immediate, though non-trivial, lift to superpolynomials.
In addition to this we discuss two extra gradings (one of them - related to that of [7] and [8]), which are implied preserve the nice factorization properties of the polynomials and can deserve further investigation.

## 2 Reinterpretation of the formula for rectangular HOMFLY

Differential expansion can be considered as a $q$-deformation of the formula for special polynomials $[13,50]$, arising from reduced HOMFLY in the limit $q=1$ : for any knot $\mathcal{K}$ and in any representation $R$

$$
\begin{equation*}
H_{R}^{\mathcal{K}}(q=1, A)=\left(H_{\square}^{\mathcal{K}}(q=1, A)\right)^{|R|} \tag{3}
\end{equation*}
$$

Moreover, in topological framing reduced HOMFLY turns into unity, when $A=q^{ \pm 1}$, what means that

$$
\begin{equation*}
H_{\square}^{\mathcal{K}}(q, A)=1+F_{\square}^{\mathcal{K}}(q, A) \cdot\{A q\}\{A / q\} \tag{4}
\end{equation*}
$$

with $\{x\}:=x-x^{-1}$ and some function $F_{\square}^{\mathcal{K}}(q, A)$, which for the figure-eight knot $\mathcal{K}=4_{1}$ is just unity. It follows that the special polynomial

$$
\begin{equation*}
H_{R}^{4_{1}}(q=1, A)=\left(1+\{A\}^{2}\right)^{|R|}=\sum_{k=0}^{|R|}\binom{|R|}{k}\{A\}^{2 k} \tag{5}
\end{equation*}
$$

Differential expansion [17, 33, 8, 35] substitutes the $q$-independent powers $\{A\}^{2 k}$ and binomial coefficients by more involved representation-dependent $k$-linear combinations of $Z$-factors $Z_{i \mid j}=\left\{A q^{i}\right\}\left\{A / q^{j}\right\}$ with $q$ dependent coefficients. The structure of this deformation strongly depends on the "defect" of the knot [35], which is miraculously regulated by the power of Alexander polynomial $H_{\mathrm{\square}}(q, A=1)$, arising from HOMFLY at $A=1$, and is not fully revealed yet.

The biggest achievement at this moment is the recently suggested in formula [36] for the differential expansion of rectangular HOMFLY polynomials for $4_{1}$

$$
\begin{gather*}
H_{\left[r^{s}\right]}^{4_{1}}=\sum_{F=0}^{\min (r, s)} \sum_{\substack{0 \leq a_{F}<\ldots<a_{3}<a_{2}<a_{1}<r \\
0 \leq b_{F}<\ldots<b_{3}<b_{2}<b_{1}<s}} \prod_{f^{\prime}<f^{\prime \prime}}^{F}\left(\frac{\left[a_{f^{\prime}}-a_{f^{\prime \prime}}\right]\left[b_{f^{\prime}}-b_{f^{\prime \prime}}\right]}{\left[a_{f^{\prime}}+b_{f^{\prime \prime}}+1\right]\left[a_{f^{\prime \prime}}+b_{f^{\prime}}+1\right]}\right)^{2} \cdot \\
\prod_{f=1}^{F}\left(h^{a_{f}+b_{f}+1}\left(\frac{\left[a_{f}+b_{f}\right]!}{\left(\left[a_{f}\right]!\left[b_{f}\right]!\right.}\right)^{2} \frac{\left[r+b_{f}\right]!\left[s+a_{f}\right]!}{\left[r-1-a_{f}\right]!\left[s-1-b_{f}\right]!\left(\left[a_{f}+b_{f}+1\right]!\right)^{2}} \prod_{\prod_{i_{f}=-b_{f}}}^{a_{f}}\left\{A q^{r+i_{f}}\right\}\left\{A q^{i_{f}-s}\right\}\right) \tag{6}
\end{gather*}
$$

and its further deformation to arbitrary twisted knots in [46]. For convenience we introduced here the forthgrading $[7,8]$ parameter $h$, which counts the number of $Z$-factors - actually, for HOMFLY $h=1$. Quantum numbers are defined as $[x]=\frac{\left\{q^{x}\right\}}{\{q\}}=\frac{q^{x}-q^{-x}}{q-q^{-1}}$.

Our primary task in this paper is to reveal the structure of this complicated expression and rewrite it in a very simple form (17).

The first observation is that the sum in (6) is actually over Young diagrams $\lambda$, which are sub-diagrams of rectangular $R=\left[r^{s}\right]$ :

$$
\begin{equation*}
H_{\left[r^{s}\right]}\left(4_{1}\right) \stackrel{(6)}{=} \sum_{\lambda \subset\left[r^{s}\right]} h^{|\lambda|} \cdot \mathcal{C}_{\lambda}^{\left[r^{s}\right]}(q) \cdot Z_{r \mid s}^{\lambda}(A, q) \tag{7}
\end{equation*}
$$

where the $A$-dependent $Z_{r \mid s}^{\lambda}$, underlined in (6), is a product of "shifted" $Z$-factors

$$
\begin{equation*}
Z_{r \mid s}^{\lambda}(A, q)=\prod_{\square \in \lambda} Z_{r \mid s}^{\left(a^{\prime}(\square)-l^{\prime}(\square)\right)}=\prod_{\square \in \lambda}\left\{A q^{r+a^{\prime}(\square)-l^{\prime}(\square)}\right\}\left\{A q^{-s+a^{\prime}(\square)-l^{\prime}(\square)}\right\} \tag{8}
\end{equation*}
$$

The second observation is that the $A$-independent coefficients $\mathcal{C}_{\lambda}$ have a very simple form. To understand this it is necessary first to return to (5) and decompose binomial coefficients into contributions of Young subdiagrams. Namely, for factorized $|R|=r s$ there is a remarkable decomposition:

$$
\begin{equation*}
\binom{r s}{k}=\frac{(r s)!}{k!(r s-k)!}=\sum_{|\lambda|=k} d_{\lambda^{t r}}(r) \cdot d_{\lambda}(s) \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
d_{\lambda}(N)=\prod_{\square \in \lambda} \frac{N-l^{\prime}(\square)+a^{\prime}(\square)}{a(\square)+l(\square)+1} \tag{10}
\end{equation*}
$$


is dimension of representations $\lambda$ of the Lie algebra $s l_{N}$, while $a, l, a^{\prime}, l^{\prime}$ are arms, legs, co-arms and co-legs, associated with a box $\square$ inside the Young diagram. The simplest particular cases of (9) are:

$$
\begin{gathered}
\binom{r s}{2}=\binom{r+1}{2}\binom{s}{2}+\binom{r}{2}\binom{s+1}{2}=d_{[2]}(r) \cdot d_{[11]}(s)+d_{[11]}(r) \cdot d_{[2]}(s) \\
\binom{r s}{3}=\binom{r+2}{3}\binom{s}{3}+4 \cdot\binom{r+1}{3}\binom{s+1}{3}+\binom{r}{3}\binom{s+2}{3}=d_{[3]}(r) \cdot d_{[111]}(s)+d_{[21]}(r) \cdot d_{[21]}(s)+d_{[111]}(r) \cdot d_{[3]}(s)
\end{gathered}
$$

and it is easy to check that at $q=1$ the sophisticated coefficients in (6) are given by exactly these formulas.
Decomposition (9) follows from the Cauchy identity in the form

$$
\begin{equation*}
\exp \left(\sum_{k=1}^{\infty} \frac{(-h)^{k} p_{k} \bar{p}_{k}}{k}\right)=\sum_{\lambda} h^{|\lambda|} \cdot \operatorname{Schur}_{\lambda^{t r}}\{p\} \cdot \operatorname{Schur}_{\lambda}\{\bar{p}\} \tag{11}
\end{equation*}
$$

for all $p_{k}=r$ and $\bar{p}_{k}=s$. Indeed, for this choice of time-variables

$$
\begin{equation*}
\operatorname{Schur}_{\lambda}\left\{\text { all } p_{k}=r\right\}=d_{\lambda}(r) \tag{12}
\end{equation*}
$$

and the identity reduces to

$$
\begin{equation*}
(1+h)^{r s}=\sum_{\lambda} h^{|\lambda|} \cdot d_{\lambda^{t r}}(r) \cdot d_{\lambda}(s) \tag{13}
\end{equation*}
$$

It is now clear what the $q$-deformation is: the quantum version of (12) is

$$
\begin{equation*}
S_{\lambda}^{*}(N \mid q)=\operatorname{Schur}_{\lambda}\left\{p_{k}=p_{k}^{*}\right\}=D_{\lambda}(N) \tag{14}
\end{equation*}
$$

where the time-variables are restricted to topological locus

$$
\begin{equation*}
p_{k}^{*}=\frac{\left\{A^{k}\right\}}{\left\{q^{k}\right\}} \stackrel{A=q^{N}}{=} \frac{[N k]}{[k]} \tag{15}
\end{equation*}
$$

and $D_{\lambda}(N)$ are quantum dimensions of representation $\lambda$ of the quantum group $U_{q}\left(s l_{N}\right)$ :

$$
\begin{equation*}
D_{\lambda}(N)=\prod_{\square \in \lambda} \frac{\left[N-l^{\prime}(\square)+a^{\prime}(\square)\right]}{[a(\square)+l(\square)+1]} \tag{16}
\end{equation*}
$$

Surprisingly or not, this quantum deformation provides the $q$-dependent coefficients in (6) and it acquires the very explicit and simple form (7):

$$
\begin{equation*}
H_{\left[r^{s}\right]}^{4_{1}}(q, A) \stackrel{(6)}{=} \sum_{\lambda \subset\left[r^{s}\right]} h^{|\lambda|} \cdot D_{\lambda^{t r}}(r) \cdot D_{\lambda}(s) \cdot Z_{r \mid s}^{\lambda}(A, q) \tag{17}
\end{equation*}
$$

with $Z$-factors (8).
Since $D_{\lambda^{t r}}(N \mid q)=D_{\lambda}\left(N, q^{-1}\right)$, while $Z_{r \mid s}^{\lambda^{t r}}(A, q)=Z_{s \mid r}^{\lambda}\left(A, q^{-1}\right)$, we get the usual symmetry property $H_{\left[r^{s}\right]}(q, A)=H_{\left[s^{r}\right]}\left(q^{-1}, A\right)$. Since $D_{\lambda}(r)$ vanishes for all $\lambda$ with more than $r$ columns, the sum in (17) is automatically restricted to $\lambda$ with no more than $r$ columns and $s$ rows, i.e. to sub-diagrams of the original $R=\left[r^{s}\right]$.

In somewhat more abstract language the Cauchy identity is the well-known decomposition of $G L(V) \times$ $G L(W)$-module

$$
\begin{equation*}
\Lambda_{h}^{*}(V \otimes W)=\sum_{\lambda} h^{|\lambda|} S_{\lambda} V \otimes S_{\lambda^{t}} W \tag{18}
\end{equation*}
$$

and the classical dimensions are just

$$
\begin{equation*}
d_{\lambda}(N)=\operatorname{dim} S_{\lambda}\left(\mathbb{C}^{N}\right) \tag{19}
\end{equation*}
$$

and if we view

$$
\begin{equation*}
x=q^{r}, \quad y=q^{s} \tag{20}
\end{equation*}
$$

as equivariant parameters and $\{x\}=x-x^{-1}$ - as the K-theoretical A-genus [9], then

$$
\begin{equation*}
H_{\left[r^{s}\right]}^{4_{1}} \stackrel{(17)}{=} 1+\sum_{\lambda} h^{|\lambda|} \prod_{\square \in \lambda} \frac{\left\{x q^{a_{\square}^{\prime}-l_{\square}^{\prime}}\right\}\left\{y q^{l_{\square}^{\prime}-a_{\square}^{\prime}}\right\}}{\left\{q_{\square}^{a_{\square}+l_{\square}+1}\right\}\left\{q_{\square}^{a_{\square}+l_{\square}+1}\right\}}\left\{A x q^{l_{\square}^{\prime}-a_{\square}^{\prime}}\right\}\left\{A y^{-1} q^{l_{\square}^{\prime}-a_{\square}^{\prime}}\right\} \tag{21}
\end{equation*}
$$

is actually the Lefshetz fixed-point formula [45], applied to a certain sheaf on the Hilbert scheme (fixed points of the torus action are labeled by Young diagrams). This means that all rectangular HOMFLY polynomials can be read out of the $(q, q)$-equivariant Euler characteristic of a certain sheaf (universal for all representations).

Now it is natural to consider $(q, t)$-equivariant characteristics, i.e. to relax the condition $q=t$ to $t=q^{\beta}$.

## $3 \beta$-deformation

The universal part of the $\beta$-deformation of differential expansions is the change of the $Z$-factors:

$$
\begin{align*}
Z_{r \mid s}^{(i \mid j)}=\left\{A q^{r+i-j}\right\}\left\{A q^{-s+i-j}\right\} \quad & \longrightarrow \quad \mathcal{Z}_{r \mid s}^{(i \mid j)}=\left\{A q^{r+i} / t^{j}\right\}\left\{A q^{i} / t^{s+j}\right\}= \\
& =(-)^{r+1} \cdot\left(\mathbf{a}^{2} \mathbf{q}^{r-s+2 i-2 j} \mathbf{t}^{r+2 i+1}+\mathbf{q}^{r+s} \mathbf{t}^{r}+\frac{1}{\mathbf{q}^{r+s} \mathbf{t}^{r}}+\frac{1}{\mathbf{a}^{2} \mathbf{q}^{r-s+2 i-2 j} \mathbf{t}^{r+2 i+1}}\right) \tag{22}
\end{align*}
$$

which, modulo an overall sign, is a positive Laurent polynomial in the DGR variables [11]

$$
\begin{equation*}
\mathbf{q}=t \quad \mathbf{t}=-q / t \quad \mathbf{a}=A \sqrt{t / q} \tag{23}
\end{equation*}
$$

For composite $Z$-factors (8) the shifts $i$ and $j$ are well defined as $a^{\prime}$ and $l^{\prime}$ respectively, and (22) implies

$$
\begin{equation*}
Z_{r \mid s}^{\lambda}(q, A) \rightarrow \mathcal{Z}_{r \mid s}^{\lambda}(q, t, A)=\prod_{\square \in \lambda} \mathcal{Z}_{r \mid s}^{\left(a_{\square}^{\prime} \mid l_{\square}^{\prime}\right)}=\prod_{\square \in \lambda}\left\{A q^{r+a_{\square}^{\prime} / t^{l_{\square}^{\prime}}}\right\}\left\{A q^{a_{\square}^{\prime}} t^{s+l_{\square}^{\prime}}\right\} \tag{24}
\end{equation*}
$$

Challenging is the deformation of the coefficients in front of the $Z$-factors. For figure-eight knot the suggestion of [17] was to rewrite the formula for symmetric HOMFLY in the form, where all the coefficients are just unities:

$$
\begin{equation*}
H_{[r]}^{4_{1}}(q, A)=1+\sum_{k=1}^{r} \frac{[r]!}{[k]![r-k]!} \prod_{i=0}^{k-1}\left\{A q^{r+1}\right\}\left\{A q^{i-1}\right\}=\sum_{k=0}^{r} \sum_{1 \leq i_{1}<\ldots<i_{k} \leq r} Z_{i_{1}}(A) Z_{i_{2}}(A q) \ldots Z_{i_{k}}\left(A q^{k-1}\right) \tag{25}
\end{equation*}
$$

where in this case the relevant $Z$-factors are

$$
\begin{equation*}
Z_{i}(A)=\left\{A q^{2(r-i)+1}\right\}\{A / q\} \tag{26}
\end{equation*}
$$

Since $r-i \geq 0$ they are naturally deformed $Z_{i}(A) \rightarrow \mathcal{Z}_{i}(A)=\left\{A q^{2(p-i)+1}\right\}\{A / t\}$, and the deformation (symmetric superpolynomial) is just

$$
\begin{equation*}
P_{[r]}^{4_{1}}(q, t, A)=\sum_{k=0}^{r} \sum_{1 \leq i_{1}<\ldots<i_{k} \leq r} \mathcal{Z}_{i_{1}}(A) \mathcal{Z}_{i_{2}}(A q) \ldots \mathcal{Z}_{i_{k}}\left(A q^{k-1}\right)=1+\sum_{k=1}^{r} \underbrace{\frac{[k]_{q}!}{\left[k![r-k]_{q}!\right.}}_{M_{\left[1^{k}\right]}^{t^{-1, q^{-1}}\left(A=q^{r}\right)}} \prod_{i=0}^{k-1}\left\{A q^{r+i}\right\}\left\{A q^{i} / t\right\} \tag{27}
\end{equation*}
$$

i.e. the binomial coefficients actually remain intact - the ratios of $q$-numbers, while $t$ appears only in the $\mathcal{Z}$. Conversely, for antisymmetric representations binomial coefficients are made fully from the $t$-numbers:

$$
\begin{equation*}
P_{\left[1^{s}\right]}^{4_{1}}(q, t, A)=1+\sum_{k=1}^{s} \frac{[s]_{t}!}{\underbrace{k]_{t}![s-k]_{t}!}_{M_{\left[1^{k}\right]}^{q, t}\left(A=t^{s}\right)}} \prod_{i=0}^{k-1}\left\{A / t^{s+1}\right\}\left\{A q / t^{i}\right\} \tag{28}
\end{equation*}
$$

In these formulas we showed also, that the binomial coefficients are expressible through Macdonald dimensions $M_{R}^{q, t}(A)$ - the values of Macdonald polynomials $M_{R}^{q, t}\{p\}$ at the topological locus $p_{k}=p_{k}^{*}(A, t):=\frac{\left\{A^{k}\right\}}{\left\{t^{k}\right\}}$,

$$
\begin{gather*}
M_{[1]}^{q, t}(A)=\frac{\{A\}}{\{t\}} \\
M_{[2]}^{q, t}(A)=\frac{\{A\}\{A q\}}{\{t\}\{q t\}} \quad M_{[11]}^{q, t}(A)=\frac{\{A\}\{A / t\}}{\{t\}\left\{t^{2}\right\}} \\
M_{[3]}^{q, t}(A)=\frac{\{A\}\{A q\}\left\{A q^{2}\right\}}{\{t\}\{q t\}\left\{q^{2} t\right\}} \quad M_{[21]}^{q, t}(A)=\frac{\{A\}\{A q\}\{A / t\}}{\{t\}^{2}\left\{q t^{2}\right\}} \quad M_{[111]}^{q, t}(A)=\frac{\{A\}\{A / t\}\left\{A / t^{2}\right\}}{\{t\}\left\{t^{2}\right\}\left\{t^{3}\right\}} \\
M_{[4]}^{q, t}(A)=\frac{\{A\}\{A q\}\left\{A q^{2}\right\}\left\{A q^{3}\right\}}{\{t\}\{q t\}\left\{q^{2} t\right\}\left\{q^{3} t\right\}} \quad M_{[31]}^{q, t}(A)=\frac{\{A\}\{A q\}\left\{A q^{2}\right\}\{A / t\}}{\{t\}^{2}\{q t\}\left\{q^{2} t^{2}\right\}} \quad M_{[22]}^{q, t}(A)=\frac{\{A\}\{A q\}\{A / t\}\{A q / t\}}{\{t\}\left\{t^{2}\right\}\{q t\}\left\{q t^{2}\right\}} \\
M_{[211]}^{q, t}(A)=\frac{\{A\}\{A q\}\{A / t\}\left\{A / t^{2}\right\}}{\{t\}^{2}\left\{t^{2}\right\}\left\{q t^{3}\right\}} \quad M_{[1111]}^{q, t}(A)=\frac{\{A\}\{A / t\}\left\{A / t^{2}\right\}\left\{A / t^{3}\right\}}{\{t\}\left\{t^{2}\right\}\left\{t^{3}\right\}\left\{t^{4}\right\}} \tag{29}
\end{gather*}
$$

Note that sums in both expressions (27) and (28) for symmetric and antisymmetric $R=[r]$ and $R=\left[1^{s}\right]$ involve only single-row??? diagrams [1 $1^{k}$, but differ by the change $(q, t) \longrightarrow\left(t^{-1}, q^{-1}\right)$, which also applies to the values of $p_{k}^{*}$ - the change is performed directly in (29). It also deserves noting that this change is the usual ingredient of the transposition rule for Macdonald polynomials

$$
\begin{equation*}
\left.M_{\lambda^{t r}}^{q, t}\left\{p_{k}\right\}=M_{\lambda}^{t^{-1}, q^{-1}}\left(-\frac{\left\{t^{k}\right\}}{\left\{q^{k}\right\}} p_{k}\right) \cdot \prod_{\square \in \lambda}\left(-\frac{\left\{q^{l} \square t^{a} \square+1\right.}{\left\{q^{l} \square+1\right.} t^{a} \square\right\}\right) \tag{30}
\end{equation*}
$$

which substitutes the simple one for Schur functions,

$$
\begin{equation*}
\operatorname{Schur}_{\lambda^{t r}}\left\{p_{k}\right\}=(-)^{|\lambda|} \cdot \operatorname{Schur}_{\lambda}\left\{-p_{k}\right\} \quad \Longrightarrow \quad D_{\lambda^{t r}}(N \mid q)=D_{\lambda^{t r}}\left(N \mid q^{-1}\right)=(-)^{|\lambda|} \cdot D_{\lambda}(-N \mid q) \tag{31}
\end{equation*}
$$

Combining this observation with the new expression (17) for combinatorial coefficients, one can easily guess the $\beta$-deformation of all rectangular HOMFLY polynomials. In abbreviated notation the suggestion is

$$
\begin{equation*}
P_{\left[r^{s}\right]}^{4_{1}}(q, t, A)=\sum_{\lambda \subset[r]^{s}} h^{|\lambda|} \cdot \mathcal{M}_{\lambda^{t r}}^{t r}(r) \cdot \mathcal{M}_{\lambda}(s) \cdot \mathcal{Z}_{r \mid s}^{\lambda} \tag{32}
\end{equation*}
$$

with the $\mathcal{Z}$-factor from the r.h.s. of (24). More explicitly,

$$
\begin{align*}
& P_{\left[r^{s}\right]}^{4_{1}}(q, t, A)=\sum_{\lambda} h^{|\lambda|} \cdot \overbrace{M_{\lambda^{t r}}^{t^{-1}, q^{-1}}\left(A=q^{r}\right)}^{\mathcal{M}_{\lambda}^{t r}(r)} \cdot \overbrace{M_{\lambda}^{q, t}\left(A=t^{s}\right)}^{\mathcal{M}_{\lambda}(s)} \cdot \mathcal{Z}_{r \mid s}^{\lambda}= \\
& =\sum_{k=1}^{p} h^{|\lambda|} \cdot M_{\lambda^{t r}}^{t^{-1}, q^{-1}}\left(p_{i}=\frac{\left\{q^{-r i}\right\}}{\left\{q^{-i}\right\}}\right) \cdot M_{\lambda}^{q, t}\left(p_{i}=\frac{\left\{t^{s i}\right\}}{\left\{t^{i}\right\}}\right) \cdot \mathcal{Z}_{r \mid s}^{\lambda}= \\
& =1+\sum_{\lambda \subset\left[r^{s}\right]} h^{|\lambda|} \underbrace{\left.\prod_{\square \in \lambda} \frac{\left\{q^{r-a_{\square}^{\prime}} t^{l^{\prime}} \square\right\}}{\left\{t^{l} \square q^{a_{\square}+1}\right\}} \frac{\left\{t^{s-l_{\square}^{\prime}} q^{a_{\square}^{\prime}}\right.}{\left\{t^{l} \square^{+1}\right.} q^{a_{\square}}\right\}}_{\text {contr }_{\lambda}}\left\{A q^{r+a_{\square}^{\prime}} t^{-l_{\square}^{\prime}}\right\}\left\{A q^{a_{\square}^{\prime}} t^{-s-l_{\square}^{\prime}}\right\} \tag{33}
\end{align*}
$$

The sign of the $\mathcal{Z}$ factor, originating from $(-)^{r+1}$ in (22), is compensated by exactly the same sign, arising in $\mathcal{M}_{\lambda^{t r}}^{t r}(r)$ after the change (23). A more serious problem could be that, in variance with quantum dimensions $D_{\lambda}(N \mid q)$, Macdonald dimensions $\mathcal{M}_{\lambda}(N \mid q)$ are not Laurent polynomials, even for concrete integer values of $N$. Surprisingly or not, however, the numerators disappear after summation over all sub-diagrams $\lambda$ (actually, they do so in every order in $|\lambda|$ ), and (33) is always (Laurent) polynomial and positive in the DGR variables (23)! Eq.(33) reproduces all previously suggested formulas for colored super- and hyper- polynomials of $4_{1}$ (and also $3_{1}$ ), with the single exception of that in [26] (which, however, deviates already from the conventional answer for the fundamental representation with $r=s=1$ ).

## 4 Polynomiality

To demonstrate how polynomiality emerges it deserves providing a couple of examples.
In the case of representation $R=[2]$ contributing are just three sub-diagrams

$$
\begin{aligned}
& \operatorname{contr}_{[\mathrm{l}]}=1 \\
& \operatorname{contr}_{[1]}=[2]_{q}\left\{A q^{2}\right\}\{A / t\} \rightarrow \mathbf{a}^{2} \mathbf{q}^{2} \mathbf{t}^{4}+\frac{1}{\mathbf{a}^{2} \mathbf{q}^{2} \mathbf{t}^{4}}+\mathbf{a}^{2} \mathbf{t}^{2}+\frac{1}{\mathbf{a}^{2} \mathbf{t}^{2}}+\mathbf{q}^{4} \mathbf{t}^{3}+\frac{1}{\mathbf{q}^{4} \mathbf{t}^{3}}+\mathbf{q}^{2} \mathbf{t}+\frac{1}{\mathbf{q}^{2} \mathbf{t}} \\
& \operatorname{contr}_{[2]}=\left\{A q^{2}\right\}\{A / t\}\left\{A q^{3}\right\}\{A q / t\} \rightarrow \\
& \rightarrow \mathbf{a}^{4} \mathbf{q}^{4} \mathbf{t}^{8}+\frac{1}{\mathbf{a}^{4} \mathbf{q}^{4} \mathbf{t}^{8}}+\mathbf{a}^{2} \mathbf{q}^{6} \mathbf{t}^{7}+\frac{1}{\mathbf{a}^{2} \mathbf{q}^{6} \mathbf{t}^{7}}+\mathbf{a}^{2} \mathbf{q}^{4} \mathbf{t}^{5}+\frac{1}{\mathbf{a}^{2} \mathbf{q}^{4} \mathbf{t}^{5}}+\frac{\mathbf{a}^{2} \mathbf{t}}{\mathbf{q}^{2}}+\frac{\mathbf{q}^{2}}{\mathbf{a}^{2} \mathbf{t}}+\mathbf{a}^{2} \mathbf{t}^{3}+\frac{1}{\mathbf{a}^{2} \mathbf{t}^{3}}+\mathbf{q}^{6} \mathbf{t}^{4}+\frac{1}{\mathbf{q}^{6} \mathbf{t}^{4}}+\mathbf{q}^{2} \mathbf{t}^{2}+\frac{1}{\mathbf{q}^{2} \mathbf{t}^{2}}+2
\end{aligned}
$$

There is exactly one diagram in each order $|\lambda|$, thus each of the tree contributions is per se a positive polynomial.
In the case of representation $R=[2,2]$ the number of contributing sub-diagrams is already six:

$$
\text { contr }_{[1]}=1
$$

$$
\begin{aligned}
& \operatorname{contr}_{[1]}=[2]_{q}[2]_{t}\left\{A q^{2}\right\}\left\{A / t^{2}\right\} \rightarrow \\
& \rightarrow \mathbf{a}^{2} \mathbf{q}^{2} \mathbf{t}^{4}+\frac{1}{\mathbf{a}^{2} \mathbf{q}^{2} \mathbf{t}^{4}}+\frac{\mathbf{q}^{2}}{\mathbf{a}^{2} \mathbf{t}^{2}}+\frac{\mathbf{a}^{2} \mathbf{t}^{2}}{\mathbf{q}^{2}}+\mathbf{a}^{2} \mathbf{t}^{4}+\frac{1}{\mathbf{a}^{2} \mathbf{t}^{4}}+\mathbf{a}^{2} \mathbf{t}^{2}+\frac{1}{\mathbf{a}^{2} \mathbf{t}^{2}}+\mathbf{q}^{6} \mathbf{t}^{3}+\frac{1}{\mathbf{q}^{6} \mathbf{t}^{3}}+\mathbf{q}^{4} \mathbf{t}^{3}+\frac{1}{\mathbf{q}^{4} \mathbf{t}^{3}}+\mathbf{q}^{4} \mathbf{t}+\frac{1}{\mathbf{q}^{4} \mathbf{t}}+\mathbf{q}^{2} \mathbf{t}+\frac{1}{\mathbf{q}^{2} \mathbf{t}}
\end{aligned}
$$

$$
\begin{aligned}
& \text { contr }_{[2]}=[2]_{t} \frac{\left\{q t^{2}\right\}}{\{q t\}}\left\{A q^{3}\right\}\left\{A q^{2}\right\}\left\{A q / t^{2}\right\}\left\{A / t^{2}\right\} \rightarrow \\
& \frac{\left(\mathbf{q}^{2}+1\right)\left(\mathbf{q}^{3} \mathbf{t}-1\right)\left(\mathbf{q}^{3} \mathbf{t}+1\right)\left(\mathbf{a}^{2} \mathbf{t}+\mathbf{q}^{4}\right)\left(\mathbf{a}^{2} \mathbf{t}^{3}+\mathbf{q}^{2}\right)\left(\mathbf{a}^{2} \mathbf{q}^{4} \mathbf{t}^{5}+1\right)\left(\mathbf{a}^{2} \mathbf{q}^{6} \mathbf{t}^{7}+1\right)}{\mathbf{a}^{4} \mathbf{q}^{10} \mathbf{t}^{8}\left(\mathbf{q}^{2} \mathbf{t}-1\right)\left(\mathbf{q}^{2} \mathbf{t}+1\right)} \\
& \text { contr }_{[1,1]}=[2]_{q} \frac{\left\{q^{2} t\right\}}{\{q t\}}\left\{A q^{2}\right\}\left\{A q^{2} / t\right\}\left\{A / t^{2}\right\}\left\{A / t^{3}\right\} \rightarrow \\
& \frac{\left(\mathbf{q}^{2} \mathbf{t}^{2}+1\right)\left(\mathbf{q}^{3} \mathbf{t}^{2}-1\right)\left(\mathbf{q}^{3} \mathbf{t}^{2}+1\right)\left(\mathbf{a}^{2} \mathbf{t}+\mathbf{q}^{4}\right)\left(\mathbf{a}^{2} \mathbf{t}+\mathbf{q}^{6}\right)\left(\mathbf{a}^{2} \mathbf{q}^{2} \mathbf{t}^{5}+1\right)\left(\mathbf{a}^{2} \mathbf{q}^{4} \mathbf{t}^{5}+1\right)}{\mathbf{a}^{4} \mathbf{q}^{10} \mathbf{t}^{8}\left(\mathbf{q}^{2} \mathbf{t}-1\right)\left(\mathbf{q}^{2} \mathbf{t}+1\right)} \\
& \text { contr }_{[2,1]}=[2]_{q}[2]_{t}\left\{A q^{2}\right\}\left\{A / t^{2}\right\}\left\{A q^{3}\right\}\left\{A q / t^{2}\right\}\left\{A q^{2} / t\right\}\left\{A / t^{3}\right\} \rightarrow \\
& \rightarrow \frac{\left(\mathbf{q}^{2}+1\right)\left(\mathbf{q}^{2} \mathbf{t}^{2}+1\right)\left(\mathbf{a}^{2} \mathbf{t}+\mathbf{q}^{4}\right)\left(\mathbf{a}^{2} \mathbf{t}+\mathbf{q}^{6}\right)\left(\mathbf{a}^{2} \mathbf{t}^{3}+\mathbf{q}^{2}\right)\left(\mathbf{a}^{2} \mathbf{q}^{2} \mathbf{t}^{5}+1\right)\left(\mathbf{a}^{2} \mathbf{q}^{4} \mathbf{t}^{5}+1\right)\left(\mathbf{a}^{2} \mathbf{q}^{6} \mathbf{t}^{7}+1\right)}{\mathbf{a}^{6} \mathbf{q}^{14} \mathbf{t}^{12}} \\
& \text { contr }_{[2,2]}=\left\{A q^{2}\right\}\left\{A / t^{2}\right\}\left\{A q^{3}\right\}\left\{A q / t^{2}\right\}\left\{A q^{2} / t\right\}\left\{A / t^{3}\right\}\left\{A * q^{3} / t\right\}\left\{A q / t^{3}\right\} \rightarrow \\
& \rightarrow \frac{1}{\mathbf{a}^{8} \mathbf{t}^{16} \mathbf{q}^{16}}\left(\mathbf{a}^{2} \mathbf{t}+\mathbf{q}^{4}\right)\left(\mathbf{a}^{2} \mathbf{t}+\mathbf{q}^{6}\right)\left(\mathbf{a}^{2} \mathbf{t}^{3}+\mathbf{q}^{2}\right)\left(\mathbf{a}^{2} \mathbf{t}^{3}+\mathbf{q}^{4}\right)\left(\mathbf{a}^{2} \mathbf{q}^{2} \mathbf{t}^{5}+1\right)\left(\mathbf{a}^{2} \mathbf{q}^{4} \mathbf{t}^{5}+1\right)\left(\mathbf{a}^{2} \mathbf{q}^{4} \mathbf{t}^{7}+1\right)\left(\mathbf{a}^{2} \mathbf{q}^{6} \mathbf{t}^{7}+1\right)
\end{aligned}
$$

Two of them have the same size $|[2]|=|[1,1]|$, and their individual contributions have non-positive factors both in the denominators and the numerators. However, when added, they provide a positive polynomial. Moreover, they still produce a polynomial, if added with the coefficients $c_{[2]}$ and $c_{[11]}$, provided

$$
\begin{equation*}
c_{[2]}-c_{[11]} \sim\{q t\} \tag{34}
\end{equation*}
$$

## 5 Plethystic logarithm of the series

All rectangular figure-eight superpolynomials are made from a single series:

$$
\begin{align*}
& K^{4_{1}}:=1+ \sum_{\lambda} h^{|\lambda|} \prod_{\square \in \lambda} \frac{\left\{x q^{-a_{\square}^{\prime}}\left\{t_{\square}^{l} \square t^{\prime} q^{a_{\square}+1}\right\}\right.}{} \frac{\left\{y t^{-l_{\square}^{\prime}} q^{a_{\square}^{\prime}}\left\{t_{\square}^{l^{+1}} q^{a} \square\right\}\right.}{}\left\{A x q^{a_{\square}^{\prime}} t^{-l_{\square}^{\prime}}\right\}\left\{A q^{a_{\square}^{\prime}} y^{-1} t^{-l_{\square}^{\prime}}\right\}= \\
&=1+h \cdot \frac{\{x\}\{y\}}{\{q\}\{t\}}\{A x\}\{A / y\}+h^{2} \cdot\left(\frac{\{x\}\{x / q\}\{y\}\{y q\}}{\{q\}\left\{q^{2}\right\}\{t\}\{q q\}}\{A x\}\{A x q\}\{A / y\}\{A q / y\}+\right. \\
&\left.+\frac{\{x\}\{x t\}\{y\}\{y / t\}}{\{t\}\left\{t^{2}\right\}\{q\}\{q t\}}\{A x\}\{A x / t\}\{A / y\}\{A /(y t)\}\right)+O\left(h^{3}\right) \tag{35}
\end{align*}
$$

Superpolynomials can be obtained by the specialization

$$
\begin{equation*}
P_{\left[r^{s}\right]}^{4_{1}}=K\left(x=q^{r}, y=t^{s}\right) \tag{36}
\end{equation*}
$$

It turns out that this series is a symmetric power of a more simple series $L$ :

$$
\begin{equation*}
K(x, y, A, q, t, h)=S y m^{*}(L):=\exp \left(\sum_{d=1}^{\infty} \frac{L\left(x^{d}, y^{d}, A^{d}, q^{d}, t^{d}, h^{d}\right)}{d}\right) \tag{37}
\end{equation*}
$$

If one opens the brackets $\{\ldots\}$, expansion of $K$ up to $h^{3}$ involves more than 900 items. At the some time its plethystic logarithm $L$ is much simpler:

$$
\begin{gather*}
L^{4_{1}}=\frac{\{x\}\{y\}\{A x\}\{A / y\}}{\{q\}\{t\}} \cdot\left\{h-h^{2} \cdot\left(\alpha+\alpha^{-1}\right)+\right.  \tag{38}\\
\left.+h^{3} \cdot\left(\alpha^{2}\left(q^{2}+t^{-2}-x^{2}-y^{-2}\right)-\frac{\alpha q}{t}\left(x^{-2}+y^{2}\right)+1-\frac{t}{\alpha q}\left(x^{2}+y^{-2}\right)+\alpha^{-2}\left(q^{-2}+t^{2}-x^{-2}-y^{2}\right)\right)-O\left(h^{4}\right)\right\}
\end{gather*}
$$

where $\alpha=A^{2} q / t$. It would be interesting to find the meaning and the general term of this new expansion.

## 6 Rectangular superpolynomials for the trefoil

Figure-eight is the simplest representative of the family of twist knots, which all have defect zero and thus possess a comparably simple differential expansion. The difference between twist knots is that the contribution of each diagram $\lambda \subset R$ contains an additional factor $F_{\lambda}$, which was found for the (anti)symmetric representations in [33], and extended to arbitrary rectangular HOMFLY just recently in [46].

The $\beta$-deformation of the $F$-factors is a separate problem, but in the particular case of the trefoil $3_{1}-$ another member of the twist knot - family it is simple. In this case the factors $F_{\lambda}^{3_{1}}$ and their $\beta$-deformations are no more than simple monomials. The answer is actually known for symmetric representations since [13, 17] and [33]. and generalization of our newly-discovered (32) to the case of $3_{1}$ is

$$
\begin{equation*}
P_{\left[r^{s}\right]}^{3_{1}}(q, t, A)=\sum_{\lambda \subset[r]^{s}} h^{|\lambda|} \cdot \underbrace{\left(-A^{2} q / t\right)^{|\lambda|}\left(\prod_{\square \in \lambda} q^{2 a_{\square}} t^{-2 l_{\square}}\right)}_{\mathcal{F}_{\lambda}^{3_{1}}} \cdot \mathcal{M}_{\lambda^{t r}}^{t r}(r) \cdot \mathcal{M}_{\lambda}(s) \cdot \mathcal{Z}_{r \mid s}^{\lambda} \tag{39}
\end{equation*}
$$

Like in the case of $4_{1}$ this formula produces positive Laurent polynomial - despite particular items in the sum are non-polynomial. In the example of sec. 4 this is because the condition (34) is fulfilled by the factors $c_{[2]}=\mathcal{F}_{[2]}^{3_{1}}$ and $c_{[11]}=\mathcal{F}_{[11]}^{3_{1}}$. Criteria of this kind can be used to define the $\beta$-deformation of the $F$-factors from [46] for all other twist knots.

Important thing is that $3_{1}$ is not only twist, but also a torus knot, thus this result can be compared with the torus hyperpolynomials from [14] and their 4-grading generalizations [7], which for rectangular representations $R$ are believed to provide the true superpolynomials (i.e. should coincide with the future calculation in Khovanov's approach). The result of this comparison is positive: (39) reproduces Cherednik's polynomials [14] and their generalizations. In Cherednik's case it is sufficient to substitute $A^{2} \longrightarrow-A^{2}$. To reproduce quadruply graded knot homologies of $[7]$ for $3_{1}$ and $4_{1}$, one inserts an additional parameter $\sigma$ into the differentials, i.e. further deforms the $Z$-factors, leaving the coefficients intact:

$$
\begin{equation*}
\mathcal{P}_{\left[r^{s}\right]}^{3_{1}}=\sum_{\lambda \subset[r]^{s}} h^{|\lambda|} \cdot \underbrace{\left(-A^{2} q / t\right)^{|\lambda|}\left(\prod_{\square \in \lambda} q^{2 a a^{-2 l}} t^{-2 \square^{\prime}}\right)}_{\mathcal{F}_{\lambda}^{33_{1}}} \cdot \mathcal{M}_{\lambda^{t r}}^{t r}(r) \cdot \mathcal{M}_{\lambda}(s) \cdot \prod_{\square \in \lambda}\left\{A q^{r+a_{\square}^{\prime}} / \sigma t^{l^{\prime}} \square\right\}\left\{A q^{a_{\square}^{\prime}} \sigma / t^{s+l_{\square}^{\prime}}\right\} \tag{40}
\end{equation*}
$$

and then make the change of variables:

$$
\begin{equation*}
\sigma \rightarrow \mathbf{t}_{r}^{-s}, \quad q \rightarrow-\mathbf{q} \mathbf{t}_{c}, \quad t \rightarrow \mathbf{q} / \mathbf{t}_{r}, \quad A \rightarrow \mathbf{a} \sqrt{-\mathbf{t}_{r} \mathbf{t}_{c}} \tag{41}
\end{equation*}
$$

(in these last formulas indices $r$ and $c$ are original notation of [7], this $r$ has nothing to do with the diagram $R=r^{s}$, however, the exponent $s$ in the substitute of $\sigma$ is the number of columns in $R$ ). In the case of $3_{1}$ this extends the original suggestion of [8] from (anti)symmetric to all rectangular representations. In the case of $4_{1}$ there are no formulas in [7] beyond (anti)symmetric case, thus (40) with eliminated $F$-factor, $\mathcal{F}_{\lambda}^{4_{1}}=1$ is only a conjecture.

## 7 Factorization properties

Despite this is not a priori requested, rectangular superpolynomial (33) has the following algebraic properties, which generalize factorization rules for HOMFLY at roots of unity [19, 51]:

$$
\begin{array}{lcl}
\text { at } q=1: & P_{\left[r^{s}\right]}=\left(P_{\left[1^{s}\right]}\right)^{r} & \text { for all } h, A \text { and } t \\
\text { at } t=1: & P_{\left[r^{s}\right]}=\left(P_{[r]}\right)^{s} & \text { for all } h, A \text { and } q  \tag{42}\\
\text { at } q^{2 n}=1: & P_{\left[r^{s}\right]} \cdot P_{\left[n^{s}\right]}=P_{\left[(r+n)^{s}\right]} & \text { for all } h, A \text { and } t \\
\text { at } t^{2 m}=1: & P_{\left[r^{s}\right]} \cdot P_{\left[r^{m}\right]}=P_{\left[r^{s+m}\right]} & \text { for all } h, A \text { and } q
\end{array}
$$

The difference relations from $[17,19,8]$ are also generalized - to 5 -graded polynomials $\mathcal{S}(A, q, t, \sigma, h)$ :

$$
\begin{array}{rll}
\mathcal{S}_{\left[r_{1}^{s}\right]}-\mathcal{S}_{\left[r_{2}^{s}\right]} & \text { is divisible by } & h\left\{A \sigma / t^{s}\right\}\left\{A q^{r_{1}+r_{2}} / \sigma\right\} \\
\mathcal{S}_{\left[r^{s_{1}}\right]}-\mathcal{S}_{\left[r^{s_{2}}\right]} & \text { is divisible by } & h\left\{A q^{r} / \sigma\right\}\left\{A \sigma / t^{s_{1}+s_{2}}\right\} \tag{43}
\end{array}
$$

Also, in the infinitesimal vicinity of the point $q=t=1$ we have for our rectangular superpolynomials:

$$
\begin{align*}
& \left.\frac{d\left(H_{R}-H_{[1]}^{|R|}\right)}{d q}\right|_{q=t=1}=\nu_{R} \sigma_{1}^{|R|-2} \sigma_{2} \\
& \left.\frac{d\left(H_{R}-H_{[1]}^{|R|}\right)}{d t}\right|_{q=t=1}=-\nu_{R^{t}} \sigma_{1}^{|R|-2} \sigma_{2} \tag{44}
\end{align*}
$$

where

$$
\begin{equation*}
\nu_{\lambda}=\sum_{i} \lambda_{i}\left(\lambda_{i}-1\right), \tag{45}
\end{equation*}
$$

and $\sigma_{1}, \sigma_{2}$ are the lowest special polynomials. In particular,

$$
\begin{align*}
& \left.\frac{d\left(H_{\left[1^{s}\right]}-H_{[1]}^{s}\right)}{d q}\right|_{q=t=1}=0 \\
& \left.\frac{d\left(H_{[r]}-H_{[1]}^{r}\right)}{d t}\right|_{q=t=1}=0 \tag{46}
\end{align*}
$$

Eqs.(44) validate the conjecture (15) of [52] for all rectangular representations $R$, at least in the case of the figure-eight knot.

## 8 Conclusion

In this paper we reported a substantial new space in the construction of colored superpolynomials: the answer (32) is suggested for the figure-eight knot, which is non-torus. The suggestion is for all rectangular representations $R=\left[r^{s}\right]$ and it is always positive. It reproduces the previous suggestions [11, 17, 20, 7, 29] for symmetric and antisymmetric representations $[r]$ and $\left[1^{s}\right]$ - on which there seems to be a consensus in the literature (with the single exception of [26]).

The answer (17) is an immediate (most naive) deformation of the recently suggested [36] formula for rectangular HOMFLY - after it is rewritten in the elegant form (17), which is another achievement of this paper. It has an amusingly suggestive structure - which, however, is specific for rectangular representations and remains to be better understood.

Further generalizations to non-rectangular representations and other knots should follow. Especially hopeful is the situation with twisted knots in rectangular representations, where HOMFLY was recently found in [46]. All these suggestions are based on the study of differential expansion [17, 33, 8, 35], which strongly depends on the defect of the knot [35]. Thus the natural sequence of steps would be to first look at twisted knots, then at other knots with defects zero and minus one - and then proceed to knots with positive defects, including the torus ones, for which the hyperpolynomials were suggested by I.Cherednik [14]. These torus hyperpolynomials are also positive in rectangular representations and thus have good chances to be rectangular superpolynomials - thus they can be compared with the implications of differential expansion, when they will be found. There is, however, a considerable work to do in this direction.

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