The number π and a summation by $SL(2,\mathbb{Z})$

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The sum (resp. the sum of the squares) of the defects in the triangle inequalities for the area one lattice parallelograms in the first quadrant has a surprisingly simple expression.

Namely, let $f(a,b,c,d)=\sqrt{a^2+b^2}+\sqrt{c^2+d^2}-\sqrt{(a+c)^2+(b+d)^2}$. Then,

$$\sum f(a,b,c,d)^2 = 2-\pi/2, \tag{Ж}$$

$$\sum f(a,b,c,d) = 2, \tag{ж}$$

where the sum runs by all $a,b,c,d\in\mathbb{Z}_{\geq 0}$ such that $ad-bc=1$.

We tease the reader to prove these formulae on their own. We are happy to discuss your ideas personally.

1 History: geometric approach to π

> What good your beautiful proof on the transcendence of π: why investigate such problems, given that irrational numbers do not even exists?
>
> Apocryphally attributed to Leopold Kronecker by Ferdinand Lindemann

Digit computation of π, probably, is one of the oldest research directions in mathematics. Due to Archimedes we may consider the inscribed and superscribed equilateral polygons for the unit circle. Let q_n (resp., Q_n) be the perimeter of such an inscribed (resp., superscribed) $3\cdot 2^n$-gon. The sequences $\{q_n\},\{Q_n\}$ obey the recurrence

$$Q_{n+1}=\frac{2q_nQ_n}{q_n+Q_n}, q_{n+1}=\sqrt{q_nQ_{n+1}}$$

and both converge to 2π. However this gives no closed formula.

One of the major breakthrough in studying of π was made by Euler, Swiss-born (Basel) German-Russian mathematician. In his Saint-Petersburg Academy of Science talk (December 5, 1735) and, then, paper [2], he calculated (*literally*) that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}. \tag{1}$$

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Euler's idea of a proof was to use the identity

$$1 - \frac{z^2}{6} + \cdots = \frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2\pi^2}\Big),$$

where the first equality is the Taylor series and the second equality happens because these two functions have the same set of zeroes. Equating the coefficient behind z^2 we get (1). This reasoning was not justified until Weierstrass, but there appeared many other proofs. A nice exercise to get (1) is by considering the residues of $\frac{\cot(\pi z)}{z^2}$.

2 Questions

One may ask what happens for other powers of $f(a, b, c, d)$. There is a partial answer in degree 3, which also reveals the source of our formulae.

For every primitive vector w consider a tangent line to D^2 consisting of all points p satisfying $w \cdot p + |w| = 0$. Consider a piecewise linear function $F : D^2 \to \mathbb{R}$ given by

$$F(p) = \inf_{w \in \mathbb{Z}^2 \setminus 0} (w \cdot p + |w|). \tag{2}$$

One can prove the following lemma.

Lemma 1. $4 - 2 \sum f(a, b, c, d)^3 = 3 \int_{D^2} F$.

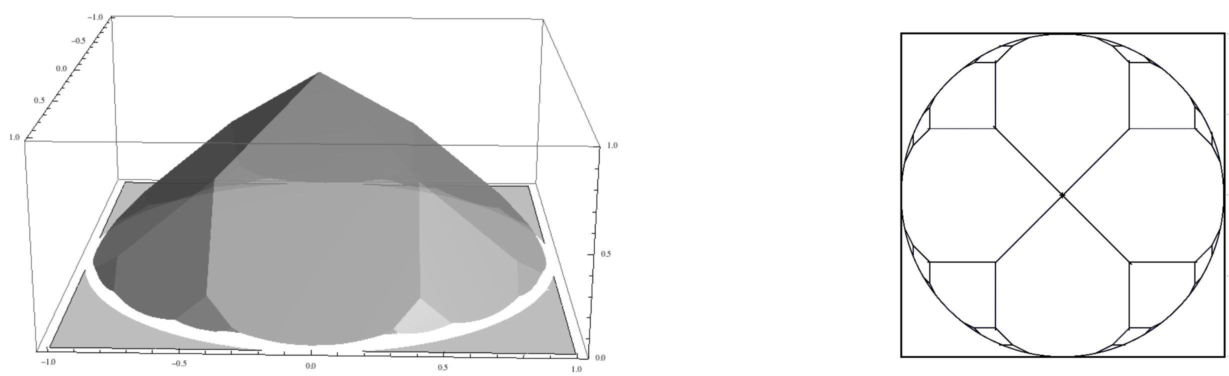

Figure 1: The plot of F and its corner locus (tropical analytic curve) C for a disc.

Now we describe the general idea behind. Denote by $C \subset D^\circ$ the locus of all points p where the function F is not smooth. The set C is a locally finite tree (see Figure 1). In fact, it is naturally a tropical curve (see [3, 4]). The numbers $f(a, b, c, d)$ represent the values of F at the vertices of C and can be computed from the equations of tangent lines.

Below we list some direction which we find intriguing to explore.

Coordinates on the space of compact convex domains. For every compact convex domain Ω we can define F_Ω as the infimum of all support functions with integral slopes, exactly as in (2). Consider the values of F_Ω at the vertices of C_Ω, the corner locus of F_Ω. These values are the complete coordinates on the set of convex domains, therefore the characteristics of Ω, for example, the area, can be potentially expressed in terms of these values. How to relate these coordinates of Ω with those of the dual domain Ω^*?

Higher dimensions. We failed to even guess similar formulae for three-dimensional bodies, but it seems that we need to sum up by all quadruples of vectors v_1, v_2, v_3, v_4 such that $\mathrm{ConvHull}(0, v_1, v_2, v_3, v_4)$ contains no lattice points.

Zeta function. We may consider the sum $\sum f(a,b,c,d)^\alpha$ as an analog of the Riemann zeta function. It converges exactly when $\alpha > 1/2$.

Other proofs. It would be nice to reprove (Ж),(ж) with other methods which are used to prove (1). Note that the vectors $(a,b),(c,d)$ can be uniquely reconstructed by the vector $(a+c, b+d)$ and our construction resembles the Farey sequence a lot. Can we interpret $f(a,b,c,d)$ as a residue of a certain function at $(a+b)+(c+d)i$? The Riemann zeta function is related to integer numbers, could it be that f is related to the Gauss integer numbers?

Modular forms. We can extend f to the whole $SL(2,\mathbb{Z})$. If both vectors $(a,b),(c,d)$ belong to the same quadrant, we use the same definition. For $(a,b),(c,d)$ from different quadrant we could define

$$f(a,b,c,d) = \sqrt{a^2+b^2} + \sqrt{c^2+d^2} - \sqrt{(a-c)^2+(b-d)^2}.$$

Then

$$\sum_{m\in SL(2,\mathbb{Z})} f(m) = \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ ad-bc=1}} f(a,b,c,d)$$

is well defined. Can we naturally extend this function to the $\mathbb{C}/SL(2,\mathbb{Z})$? Can we make similar series for other lattices or tessellations of the plane?

Aknowledgement. We want to thank the God and the universe for these beautiful formulae.

References

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