Discrete Painlevé equations and Quantum Minimal Surfaces

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References

The question we consider was posed in

J. Arnlind, J. Hoppe and M. Kontsevich, *Quantum minimal surfaces*, arXiv:1903.10792v1

and I found out about it from M. Kontsevich during his visit to BIMSA in Spring 2023. This talk is based on

P.A. Clarkson, A. Dzhamay, A.N.W. Hone, and B. Mitchell, *Special solutions of a discrete Painlevé equation for quantum minimal surfaces* arXiv:2503.14436, Theoretical and Mathematical Physics, 224(2): 1359–1397 (2025)

There are also two recent preprints by J. Hoppe and J. Hoppe and G. Felder,

J. Hoppe, Quantum states from minimal surfaces, arXiv:2502.18422

G. Felder and J. Hoppe, Orthogonal polynomials with complex densities and quantum minimal surfaces, arXiv:2504.06197

Minimal Surfaces and Quantization

Minimal surfaces are maps $\mathbf{x} : \Sigma \to \mathbb{R}^d$ that extremise the Schild functional

$$S[\mathbf{x}] = \int_\Sigma \sum_{j<k} \{x_j, x_k\}^2 \, \omega.$$

Here Σ is a surface with a symplectic form ω and associated Poisson bracket $\{\cdot, \cdot\}$, and $(x_j)_{j=1,\ldots,d}$ are coordinates on $\mathbb{R}^d$.

The Euler-Lagrange equations obtained from the action S are

$$\sum_{j=1}^{d} \{x_j, \{x_j, x_k\}\} = 0, \qquad k = 1, \ldots, d. \tag{$*$}$$

Quantize by replacing the classical observables x_j with self-adjoint operators $X_j : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$, with commutator replacing the Poisson bracket.

Definition

Quantum minimal surface is a collection of operators $\{X_j\}$ on $\mathcal{H}$ satisfying

$$\sum_{j=1}^{d} [X_j, [X_j, X_k]] = 0, \qquad k = 1, \ldots, d. \tag{$*_q$}$$

Minimal Surfaces and Quantization

For $\mathbb{R}^4 \simeq \mathbb{C}^2$, an arbitrary analytic function f defines a solution of $(*)$ via

$$z_2 = f(z_1), \qquad z_1 = x_1 + \mathrm{i}\, x_2, \quad z_2 = x_3 + \mathrm{i}\, x_4;$$

Arnlind, Hoppe and Kontsevich considered a parabola $z_2 = z_1^2$. Its quantized version, parameterized as $Z_1 = W$, $Z_2 = W^2$, gives an operator W satisfying

$$[W^\dagger, W] + [(W^\dagger)^2, W^2] = \epsilon \mathbf{1}.$$

W acts on $\mathcal{H} = \{\, |n\rangle \, | n = 0, 1, 2, \ldots\}$ via $W\,|n\rangle = w_n\,|n+1\rangle$.
Taking the expectation $\langle n| \cdot |n\rangle$ leads to

$$v_n - v_{n-1} + v_{n+1}v_n - v_{n-1}v_{n-2} = \epsilon,$$

where $v_n = |w_n|^2 \geq 0$. Integrating, we get a *discrete Painlevé equation*

$$v_n(v_{n+1} + v_{n-1} + 1) = \epsilon\, n + \zeta, \qquad (\dagger)$$

This equation is (one of) the d-$\mathrm{P_I}$ equation(s).

The Asymptotics

Taking a semi-classical limit gives the asymptotic behavior for v_n that helps us to identify the solution.

Parameterizing the classical version of the complex parabola $z_2 = z_1^2$ in polar coordinates by $z_1 = re^{\mathrm{i}\phi}$, $z_2 = r^2 e^{2\mathrm{i}\phi}$ gives a pair of canonically conjugate (flat) coordinates $\tilde{r}, \phi$, where

$$\tilde{r} = r^4 + \frac{1}{2} r^2 - c.$$

Quantize by replacing $\tilde{r}$ by the momentum operator conjugate to $\widehat{U}$,

$$\tilde{r} \to -\mathrm{i}\,\hbar \frac{\partial}{\partial \phi},$$

where $e^{\mathrm{i}\widehat{U}} |n\rangle = |n+1\rangle$ and we identify the states $|n\rangle$ for $n \geq 0$ with the non-negative modes $e^{\mathrm{i}n\phi}$ on the circle.

Then $v_n^2 + \frac{1}{2} v_n \sim n\hbar + c$ leads to $v_n \approx \frac{1}{4}\left(\sqrt{1 + 8(n+1)\epsilon} - 1\right)$.

This agrees with the asymptotic behaviour of positive solutions of (†), both in the limit $\hbar \to 0$ with n fixed, and for $n \to \infty$ with $\hbar$ fixed, provided that the conditions $\zeta = \epsilon = 2\hbar$, $c = \hbar$ are imposed.

Statement of the problem

Find an explicit analytic solution for the initial value problem

$$v_{n+1} + v_{n-1} + 1 = \frac{\epsilon(n+1)}{v_n}, \qquad v_{-1} = 0,\ v_0 > 0$$

associated with a quantum minimal surface. In particular, require $v_n \geq 0$ and $v_n \approx \frac{1}{4}\left(\sqrt{1+8(n+1)\epsilon} - 1\right)$.

Theorem

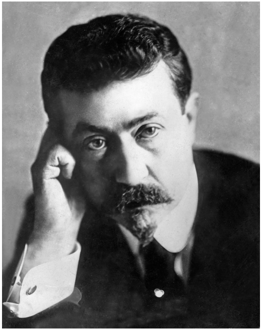
For each $\epsilon > 0$,

$$v_0 = v_0(\epsilon) = \frac{1}{2}\left(\frac{K_{5/6}(\frac{1}{2}t)}{K_{1/6}(\frac{1}{2}t)} - 1\right), \qquad \text{where} \quad t = \frac{1}{3\epsilon},$$

and $K_\nu\left(\frac{1}{2}t\right)$ is a modified Bessel function, determines the unique positive solution of this $\mathrm{dP_I}$ equation. It has the required asymptotics and for each $n \geq 0$, the corresponding quantities $v_n > 0$ are written explicitly as ratios of the Wronskian determinants whose entries are specified in terms of modified Bessel functions.

Historical Digression: Painlevé Equations

- The name *discrete Painlevé equation* is due to connections with *differential Painlevé equations*.
- Differential Painlevé equations are certain *nonlinear ODEs* that define *nonlinear special functions* (Painlevé transcendents).
- These equations satisfy the *Painlevé property* that the general solutions has no movable singularities other than the poles.



P. Painlevé (1863–1933)

- Obtained first in the work of P. Painlevé and B. Gambier, and almost simultaneously by R. Fuchs in the study of *isomonodromic deformations* of systems of *linear* ODEs.
- Very important in Mathematical Physics, Random Matrix Theory, Quantum Cohomology, Integrable Probability (Tracy-Widom distribution), etc.

Classification Scheme for Painlevé Equations

(P-I) $\frac{d^2y}{dt^2} = 6y^2 + t$; Painlevé equations have parameters! (Bäcklund symmetries)

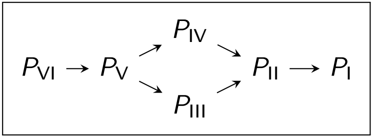
(P-II) $\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$;

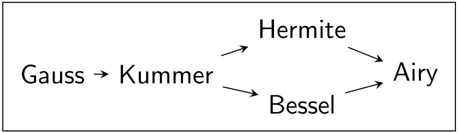
(P-III) $\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$;

(P-IV) $\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$;

(P-V) $\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$;

(P-VI) $$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).$$





The notion of discrete Painlevé Equation

- First appeared in 1990, **A.R. Its, A.V. Kitaev, and A.S. Fokas** (related works of É. Brézin and V. Kazakov and D. Gross and A. Migdal). *Matrix Models of Two-Dimensional Quantum Gravity and Isomonodromic Solutions of "Discrete Painlevé Equations"*
- Followed by 1991 paper by F. Nijhoff and V. Papageorgiou, *Similarity reductions of integrable lattices and discrete analogues of the Painlevé* II *equation* and 1992 paper by V. Papageorgiou, F. Nijhoff, B. Grammaticos, and A. Ramani, *Isomonodromic deformation problems for discrete analogues of Painlevé equations*
- Many examples were obtained in early 1990s by B. Grammaticos and A. Ramani using the *singularity confinement* criterion applied to *deautonomizations of QRT maps*.

Historic definition

Discrete Painlevé equation is a certain second-order non-autonomous recurrence relation that has a differential Painlevé equation as a continuous limit.

This is the origin of the naming conventions such as d-P_{II} or q-P_{IV}, based on the corresponding continuous limit, which is confusing.

Some Examples of Discrete Painlevé Equations

d-P_{I}: $x_{n+1}+x_n+x_{n-1}=\dfrac{an+b}{x_n}+1$;

alt. d-P_{I}: $\dfrac{an+b}{x_{n+1}+x_n}+\dfrac{a(n-1)+b}{x_n+x_{n-1}}=-x_n^2+c$;

d-P_{II}: $x_{n+1}+x_{n-1}=\dfrac{(an+b)x_n+c}{1-x_n^2}$;

alt. d-P_{II}: $\dfrac{an+b}{x_{n+1}x_n+1}+\dfrac{a(n-1)+b}{x_nx_{n-1}+1}=-x_n+\dfrac{1}{x_n}+(an+b)+c$;

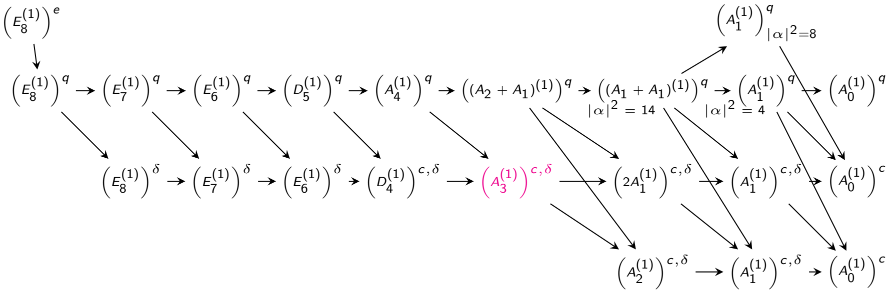
d-P_{V}: $\begin{cases} f+\bar{f}=a_3+\dfrac{a_1}{g+1}+\dfrac{a_0}{sg+1} \\ g\bar{g}=\dfrac{(\bar{f}+a_2-\delta)(\bar{f}+a_2-\delta+a_4)}{s\bar{f}(\bar{f}-a_3)} \end{cases}$, $\delta=a_0+a_1+2a_2+a_3+a_4$

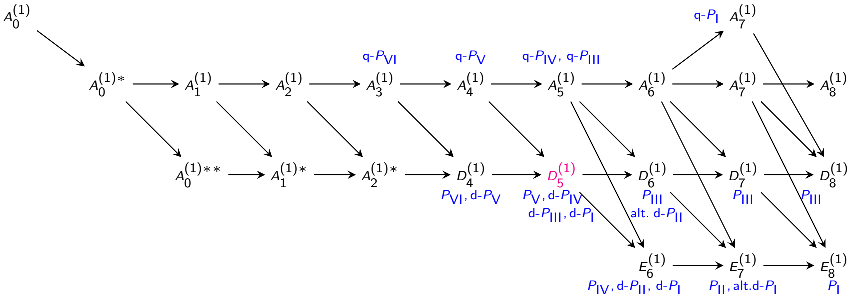
q-P_{VI}: $\begin{cases} \bar{g}=\dfrac{b_3b_4}{g}\dfrac{(f-b_5)(f-b_6)}{(f-b_7)(f-b_8)} \\ \bar{f}=\dfrac{b_7b_8}{f}\dfrac{(\bar{g}-qb_1)(\bar{g}-qb_2)}{(\bar{g}-b_3)(\bar{g}-b_4)} \end{cases}$, $\quad q=(b_3b_4b_5b_6)/(b_1b_2b_7b_8)$

The Geometric Approach (K. Okamoto, H. Sakai)

- K. Okamoto (79–87): Introduced the *Space of Initial Conditions* (SIC). For that, rewrite P_{J} as a first-order system, then the SIC is essentially the *extended phase space* on which the flow of the system is regularized.
- A *time slice* of the space is a *rational algebraic surface* $\mathcal{X}$ obtained by blowing up a complex projective plane at a number of points, and with a certain configuration of (-2) curves removed.
- This configuration of curves is the decomposition of the *polar divisor* of the symplectic form of the system into irreducible components. It can be described by an *affine Dynkin diagram* that characterizes the system (K. Takano et al.) The type of this Dynkin diagram is known as the *surface type* of our system.
- Okamoto studied symmetries (Cremona transformations) of such surfaces and showed that they give the group of Bäcklund transformations of P_{J}. These groups turned out to be some extended affine Weyl groups, e.g., for P_{VI} the group is $\widetilde{W}(D_4^{(1)})$.
- H. Sakai in *Rational surfaces associated with affine root systems and geometry of the Painlevé equations* (2001) interpreted discrete Painlevé equations as traslations in these affine Weyl groups and suggested the current classification sheme.

Sakai Classification Scheme (symmetry and surface types)





Recall our problem:

Statement of the problem

Find an explicit analytic solution for the initial value problem

$$v_{n+1} + v_{n-1} + 1 = \frac{\epsilon(n+1)}{v_n}, \qquad v_{-1} = 0,\ v_0 > 0$$

associated with a quantum minimal surface. In particular, require $v_n \geq 0$ and $v_n \approx \frac{1}{4}\left(\sqrt{1+8(n+1)\epsilon} - 1\right)$.

Theorem

For each $\epsilon > 0$,

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and $K_\nu\left(\frac{1}{2}t\right)$ is a modified Bessel function, determines the unique positive solution of this $\mathrm{dP_I}$ equation. It has the required asymptotics and for each $n \geq 0$, the corresponding quantities $v_n > 0$ are written explicitly as ratios of the Wronskian determinants whose entries are specified in terms of modified Bessel functions.

Geometry of our Discrete Painlevé Equation

We first consider a more general case of our recurrence ($\mathrm{dP_I}$ equation):

$$x_{n+1} + x_{n-1} = \frac{\tilde{\alpha} n + \tilde{\beta}}{x_n} + \tilde{\gamma},$$

which specializes to our case when $\tilde{\alpha} = \tilde{\beta} = \epsilon$ and $\tilde{\gamma} = -1$.

As usual, rewrite it as a mapping using $y_n := x_{n+1}$,

$$\varphi_n(x_n, y_n) = \left(y_n, \frac{\tilde{\alpha}(n+1) + \tilde{\beta}}{y_n} + \tilde{\gamma} - x_n \right).$$

The base points of this mapping on $\mathbb{P}^1 \times \mathbb{P}^1$ form a standard $D_5^{(1)}$ configuration:

$$q_1 \left(X = \frac{1}{x} = 0, y = \tilde{\gamma} \right) \leftarrow q_2 \left(u_1 = X = \frac{1}{x} = 0, v_1 = x(y - \tilde{\gamma}) = \tilde{\alpha} \right),$$

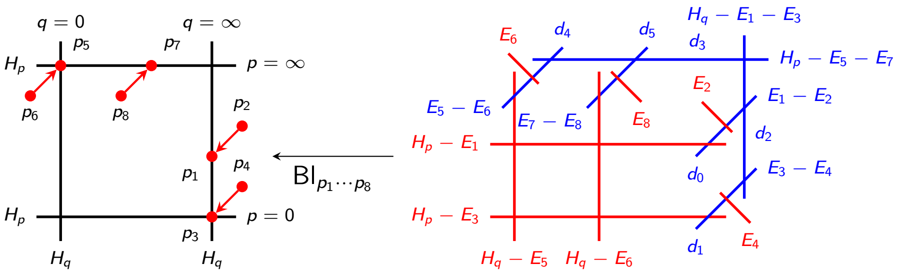
$$q_3 \left(X = \frac{1}{x} = 0, y = 0 \right) \leftarrow q_4 \left(u_3 = X = \frac{1}{x} = 0, v_3 = xy = (n+1)\tilde{\alpha} + \tilde{\beta} \right),$$

$$q_5 \left(x = 0, Y = \frac{1}{y} = 0 \right) \leftarrow q_6 \left(U_5 = xy = n\tilde{\alpha} + \tilde{\beta}, V_5 = Y = \frac{1}{y} = 0 \right),$$

$$q_7 \left(x = \tilde{\gamma}, Y = \frac{1}{y} = 0 \right) \leftarrow q_8 \left(U_7 = y(x - \tilde{\gamma}) = -\tilde{\alpha}, V_7 = Y = \frac{1}{y} = 0 \right).$$

The Standard $D_5^{(1)}$ Surface Family

The standard geometric realization of the $D_5^{(1)}$ family is $\mathcal{X} = \mathrm{Bl}_{p_1\cdots p_8}(\mathbb{P}^1 \times \mathbb{P}^1)$



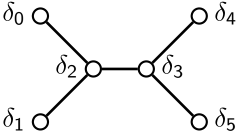
The coordinates of the basepoints are given in terms of *root variables* satisfying the usual normalization condition $a_0 + a_1 + a_2 + a_3 = 1$ by

$$p_1(\infty, -t) \leftarrow p_2(0, -a_0), \qquad p_5(0, \infty) \leftarrow p_6(a_1, 0),$$
$$p_3(\infty, 0) \leftarrow p_4(0, -a_2), \qquad p_7(1, \infty) \leftarrow p_8(a_3, 0).$$

Parameters match as $a_0 = \frac{1}{3}$, $a_1 = -\frac{n}{3} - \frac{\tilde{\beta}}{3\tilde{\alpha}}$, $a_2 = \frac{n+1}{3} + \frac{\tilde{\beta}}{3\tilde{\alpha}}$, $a_3 = \frac{1}{3}$.

Surface and Symmetry Root Bases

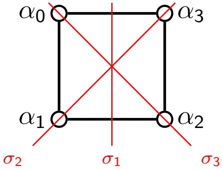
Surface $\{\delta_i = [d_i]\}$ and symmetry $\{\alpha_j\}$, where $\alpha_j \bullet \delta_i = 0$ root bases in $\mathrm{Pic}(\mathcal{X}) = \mathrm{Span}_{\mathbb{Z}}\{\mathcal{H}_x, \mathcal{H}_y, \mathcal{E}_1, \dots \mathcal{E}_8\}$ for this geometric realization are:
Surface root basis $\delta_i = [d_i]$, $\mathrm{Span}_{\mathbb{Z}}\{\delta_i\} \triangleleft \mathrm{Pic}(\mathcal{X})$ is a *surface sub-lattice*:



$$\delta_0 = \mathcal{E}_1 - \mathcal{E}_2, \qquad \delta_3 = \mathcal{H}_p - \mathcal{E}_5 - \mathcal{E}_7,$$
$$\delta_1 = \mathcal{E}_3 - \mathcal{E}_4, \qquad \delta_4 = \mathcal{E}_5 - \mathcal{E}_6,$$
$$\delta_2 = \mathcal{H}_q - \mathcal{E}_1 - \mathcal{E}_3, \qquad \delta_5 = \mathcal{E}_7 - \mathcal{E}_8.,$$

$$\delta = \delta_0 + \delta_1 + 2\delta_2 + 2\delta_3 + \delta_4 + \delta_5.$$

Symmetry root basis α_i, $\mathrm{Span}_{\mathbb{Z}}\{\alpha_i\} \triangleleft \mathrm{Pic}(\mathcal{X})$ is a *symmetry sub-lattice*:



$$\alpha_0 = \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2, \qquad \alpha_2 = \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_4,$$
$$\alpha_1 = \mathcal{H}_q - \mathcal{E}_5 - \mathcal{E}_6, \qquad \alpha_3 = \mathcal{H}_q - \mathcal{E}_7 - \mathcal{E}_8.$$

$$\delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3.$$

The symmetry group is $\widehat{W}\left(A_3^{(1)}\right) := W\left(A_3^{(1)}\right) \rtimes \mathrm{Aut}(A_3^{(1)})$.

Birational representation of $W\left(A_3^{(1)}\right)$

$$w_0 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} -a_0 & a_0 + a_1 \\ a_2 & a_0 + a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q + \dfrac{a_0}{p+t} \\ p \end{matrix} ,$$

$$w_1 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 + a_1 & -a_1 \\ a_1 + a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p - \dfrac{a_1}{q} \end{matrix} ,$$

$$w_2 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 & a_1 + a_2 \\ -a_2 & a_2 + a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q + \dfrac{a_2}{p} \\ p \end{matrix} ,$$

$$w_3 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 + a_3 & a_1 \\ a_2 + a_3 & -a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p - \dfrac{a_3}{q-1} \end{matrix} ,$$

$$\sigma_1 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_3 & a_2 \\ a_1 & a_0 \end{pmatrix} ; \ -t \ ; \begin{matrix} -\dfrac{p}{t} \\ qt \end{matrix} ,$$

$$\sigma_2 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_2 & a_1 \\ a_0 & a_3 \end{pmatrix} ; \ -t \ ; \begin{matrix} q \\ p + t \end{matrix} ,$$

$$\sigma_3 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} ; \ t \ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_0 & a_3 \\ a_2 & a_1 \end{pmatrix} ; \ -t \ ; \begin{matrix} 1 - q \\ -p \end{matrix} .$$

Examples of Discrete Painlevé Equations on $D_5^{(1)}$ Surface

- $\overline{a}_0 = a_0 + 1$, $\overline{a}_1 = a_1 - 1$, $\overline{a}_2 = a_2 + 1$, and $\overline{a}_3 = a_3 - 1$, which corresponds to the translation

$$\phi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \phi_*(\alpha) = \alpha + \langle -1, 1, -1, 1 \rangle \delta,$$

in the root lattice. We can represent it in terms of generators as $\phi = \sigma_3 \sigma_2 w_3 w_1 w_2 w_0$, and the actual equations can be written as

$$\overline{q} + q = 1 - \frac{a_2}{p} - \frac{a_0}{p+t}, \qquad p + \underline{p} = -t + \frac{a_1}{q} + \frac{a_3}{q-1}$$

- $\overline{a}_0 = a_0 + 1$, $\overline{a}_1 = a_1$, $\overline{a}_2 = a_2$, and $\overline{a}_3 = a_3 - 1$, which corresponds to the translation

$$\psi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \psi_*(\alpha) = \alpha + \langle -1, 0, 0, 1 \rangle \delta,$$

in the root lattice. We can represent it in terms of generators as $\psi = \sigma_3 \sigma_1 w_2 w_1 w_0$, and the actual equations can be written as

$$\overline{f} f = \frac{s\overline{g}}{(\overline{g} - a_3 + 1)(\overline{g} + a_0 + 1)}, \qquad \overline{g} + g = \frac{s}{f} + \frac{a_1 + a_0}{1 - f} - 1 + a_3 - a_0,$$

Which one is ours?

Linearization of the dynamics

The mapping φ induces the following action on $\mathrm{Pic}(\mathcal{X})$:

$$
\begin{aligned}
\mathcal{H}_y \xleftarrow{\varphi^*} &\mathcal{H}_x \xrightarrow{\varphi_*} \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_5 - \mathcal{E}_6,\\
\mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_3 - \mathcal{E}_4 \longleftarrow &\mathcal{H}_y \longrightarrow \mathcal{H}_x,\\
\mathcal{E}_5 \longleftarrow &\mathcal{E}_1 \longrightarrow \mathcal{E}_7,\\
\mathcal{E}_6 \longleftarrow &\mathcal{E}_2 \longrightarrow \mathcal{E}_8,\\
\mathcal{E}_7 \longleftarrow &\mathcal{E}_3 \longrightarrow \mathcal{H}_x - \mathcal{E}_6,\\
\mathcal{E}_8 \longleftarrow &\mathcal{E}_4 \longrightarrow \mathcal{H}_x - \mathcal{E}_5,\\
\mathcal{H}_y - \mathcal{E}_4 \longleftarrow &\mathcal{E}_5 \longrightarrow \mathcal{E}_1,\\
\mathcal{H}_y - \mathcal{E}_3 \longleftarrow &\mathcal{E}_6 \longrightarrow \mathcal{E}_2,\\
\mathcal{E}_1 \longleftarrow &\mathcal{E}_7 \longrightarrow \mathcal{E}_3,\\
\mathcal{E}_2 \longleftarrow &\mathcal{E}_8 \longrightarrow \mathcal{E}_4.
\end{aligned}
$$

Its action on the symmetry roots is

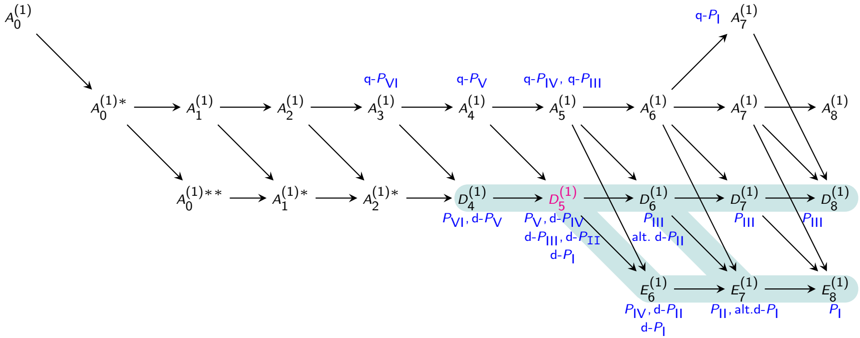
$$\varphi_* : \boldsymbol{\alpha} = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \langle \alpha_3, \alpha_0 + \alpha_1, -\alpha_1, \alpha_1 + \alpha_2 \rangle,$$

which is a *quasi-translation*: after *three* iterations we get a translation

$$\varphi_*^3 : \boldsymbol{\alpha} = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \boldsymbol{\alpha} + \langle 0, 1, -1, 0 \rangle \delta, \qquad \delta = -\mathcal{K}_{\mathcal{X}} = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3,$$

which is conjugated to the second example (Sakai's $\mathrm{dP}_{\mathrm{IV}}$ equation).

Sakai Classification Scheme (surface type)



We see that our dynamics correspond to a special parameter locus

$$a_0 = a_1 + a_2 = a_3 = \frac{1}{3}$$

for the $D_5^{(1)}$-family, and as such, this dynamic can be interpreted as Bäcklund transformations of the differential $\mathrm{P_V}$ equation for the special parameter values.

Bäcklund Transformations for $\mathrm{P_V}$

Recall the differential Painlevé V equation:

$$\frac{d^2w}{dt^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{(w-1)^2(\alpha w^2+\beta)}{t^2 w} + \frac{\gamma w}{t} - \frac{1}{2}\cdot\frac{w(w+1)}{(w-1)}.$$

In terms of root variables,

$$\alpha = \frac{a_1^2}{2}, \qquad \beta = -\frac{a_3^2}{2}, \qquad \gamma = a_0 - a_2.$$

The $D_5^{(1)}$ surface family before is the *Okamoto Space of Initial Conditions* for

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{1}{t}\Big(q(q-1)(2p+t) - a_1(q-1) - a_3 q\Big) = \dfrac{\partial H}{\partial p}, \\ \dfrac{dp}{dt} = \dfrac{1}{t}\Big(p(p+t)(1-2q) + (a_1+a_3)p - a_2 t\Big) = -\dfrac{\partial H}{\partial q}, \end{cases}$$

where

$$H(q,p;t) = \frac{1}{t}\Big(q(q-1)p(p+t) - (a_1+a_3)qp + a_1 p + a_2 tq\Big), \qquad w(t) = 1 - \frac{1}{q(t)}.$$

Bäcklund Transformations for P_{V}

Our discrete dynamics can be written as birational mappings with the help of the birational representation of $\widehat{W}\left(D_5^{(1)}\right)$:

$$\varphi = \sigma_2\sigma_1 w_2 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; \ t\ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_1+a_2 & -a_2 \\ a_2+a_3 & a_0 \end{pmatrix}; \ t\ ; \begin{matrix} -\frac{p}{t} \\ t\left(q+\frac{a_2}{p}-1\right) \end{matrix},$$

$$\varphi^{-1} = w_2\sigma_1\sigma_2 : \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; \ t\ ; \begin{matrix} q \\ p \end{matrix} \mapsto \begin{pmatrix} a_3 & a_0+a_1 \\ -a_1 & a_1+a_2 \end{pmatrix}; \ t\ ; \begin{matrix} 1+\frac{p}{t}-\frac{a_1}{qt} \\ -qt \end{matrix}.$$

This induces Bäcklund transformations

$$\varphi : w \mapsto w_+ = 1 - \frac{1}{\overline{q}} = 1 + \frac{t}{p} = 1 + \frac{2tw}{t\frac{dw}{dt} - a_1 w^2 + (a_1 - a_3 - t)w + a_3}$$

$$\varphi^{-1} : w \mapsto w_- = 1 - \frac{1}{\underline{q}} = 1 - \frac{qt}{qt+qp-a_1}$$

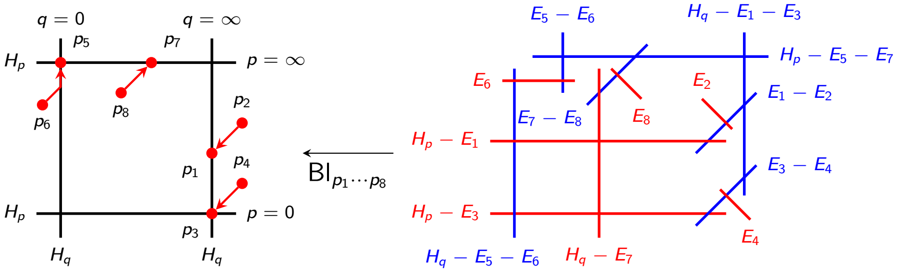
$$= 1 - \frac{2tw}{t\frac{dw}{dt} + a_1 w^2 - (a_1 + a_3 - t)w + a_3}.$$

Painlevé V and parameter specialization

It is now time to specialize to our recurrence setting $\tilde{\alpha} = \tilde{\beta} = \epsilon$ and $\tilde{\gamma} = -1$. Then

$$a_0 = a_3 = \frac{1}{3},\ a_1 = -\frac{n+1}{3},\ a_2 = \frac{n+2}{3}; \quad \alpha = \frac{(n+1)^2}{18},\ \beta = -\frac{1}{18},\ \gamma = -\frac{n+1}{3}.$$

In particular, when $n = -1$, $a_1 = 0$, which corresponds to the appearance of a nodal curve: the cascade $p_5(0,\infty) \leftarrow p_6(a_1, 0)$ changes from a generic to a *corner* point:



Riccati solution

The existence of nodal curves corresponds to reductions to Riccati equations, and this property is preserved by Bäcklund transformations.
In our case, for $n = 0$, for

$$\frac{d^2 w_0}{dt^2} = \left(\frac{1}{2w_0} + \frac{1}{w_0 - 1}\right)\left(\frac{dw_0}{dt}\right)^2 - \frac{1}{t}\frac{dw_0}{dt} + \frac{(w_0-1)^2(w_0^2-1)}{18t^2 w_0} - \frac{w_0}{3t} - \frac{w_0(w_0+1)}{2(w_0-1)}$$

the corresponding Riccati equation is

$$t\frac{dw_0}{dt} = \frac{1}{3}w_0^2 - tw_0 - \frac{1}{3}$$

which has solution

$$w_0(t) = -\frac{C_1\left\{I_{1/6}(\tfrac{1}{2}t) - I_{-5/6}(\tfrac{1}{2}t)\right\} + C_2\left\{K_{1/6}(\tfrac{1}{2}t) + K_{5/6}(\tfrac{1}{2}t)\right\}}{C_1\left\{I_{1/6}(\tfrac{1}{2}t) + I_{-5/6}(\tfrac{1}{2}t)\right\} + C_2\left\{K_{1/6}(\tfrac{1}{2}t) - K_{5/6}(\tfrac{1}{2}t)\right\}},$$

where $I_\nu(\frac{1}{2}t)$ and $K_\nu(\frac{1}{2}t)$ are modified Bessel functions, with C_1 and C_2 arbitrary constants.

Identifying the initial condition

Using $w(t) = 1 + \frac{1}{v_n(\epsilon)}$ and $t = \frac{1}{3\epsilon}$, we can then rewrite

$$v_0(t) = \frac{1}{w_0(t) - 1} = -\frac{1}{2} - \frac{C_1 I_{-5/6}(\frac{1}{2}t) - C_2 K_{5/6}(\frac{1}{2}t)}{2\{C_1 I_{1/6}(\frac{1}{2}t) + C_2 K_{1/6}(\frac{1}{2}t)\}},$$

$$v_1(t) = \frac{1}{w_1(t) - 1} = -1 - \frac{2}{3t} - \frac{2\{C_1 I_{-5/6}(\frac{1}{2}t) - C_2 K_{5/6}(\frac{1}{2}t)\}}{3t\{C_1 I_{1/6}(\frac{1}{2}t) + C_2 K_{1/6}(\frac{1}{2}t)\} + C_1 I_{-5/6}(\frac{1}{2}t) - C_2 K_{5/6}(\frac{1}{2}t)}.$$

or

$$2v_0 + 1 = \frac{K_{5/6}(\frac{1}{2}t) - \lambda I_{-5/6}(\frac{1}{2}t)}{K_{1/6}(\frac{1}{2}t) + \lambda I_{1/6}(\frac{1}{2}t)}, \qquad t = \frac{1}{3\epsilon},$$

The asymptotic requirement picks out the coefficients, i.e., $\lambda = 0$:

$$v_0(\epsilon) = \frac{1}{2}\left(\frac{K_{5/6}(\frac{1}{6\epsilon})}{K_{1/6}(\frac{1}{6\epsilon})} - 1\right)$$