Quantum invariants under the minimal model program

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Overview

We work over the complex field $\mathbb{C}$.

A variety X is called ($\mathbb{Q}$-)*Gorenstein* if K_X is ($\mathbb{Q}$-)Cartier.

A variety X is called *terminal* if K_X is $\mathbb{Q}$-Cartier and for a log resolution $\pi : Y \to X$ we can write

$$\pi^* K_X = K_Y + \sum_{E_i} a_i E_i$$

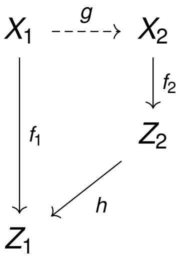
where E_i runs through exceptional divisors of π and $a_i < 0$.

A fibration $f : X \to Z$ is called *Fano* if $-K_X$ is ample over Z.

Background

Problem: Order between central models

Our fundamental object is the following diagram:



where X_1, X_2 have terminal singularities, f_1, f_2 are Fano fibrations and g is a birational 1-contraction (i.e. it doesn't extract divisors). We say that X_i/Z_i are *central models*, and the above diagram is denoted by $X_1/Z_1 \geq X_2/Z_2$.

Main problem: For a fixed central model X_1/Z_1, find all central models X_2/Z_2 such that $X_1/Z_1 \geq X_2/Z_2$.

Choi-Shokurov: There are only finitely many X_2/Z_2 up to isomorphisms.

Example 1.1 *If $\dim Cl_{\mathbb{R}}(X_1/Z_1) = 2$, then up to isomorphisms there are exactly 2 Mori fibre spaces X_2/Z_2 and X_2'/Z_2' satisfying $X_1/Z_1 \geq X_2/Z_2$ and $X_1/Z_1 \geq X_2'/Z_2'$. The birational map $X_2/Z_2 \dashrightarrow X_2'/Z_2'$ is a Sarkisov link. Every Sarkisov link can be constructed from this for some central model of rank 2.*

Example 1.2 *If $\dim Cl_{\mathbb{R}}(X_1/Z_1) = 3$, then up to isomorphisms all the central models X_2/Z_2 satisfying $X_1/Z_1 \geq X_2/Z_2$ form a circle of Sarkisov links. Such circle is called an elementary relation of Sarkisov links.*

When X_1 and X_2 are smooth Fano threefolds: Mori-Mukai, as a part of their classification results.

When X_1 is a smooth Fano threefold and X_2 is a Gorenstein terminal Fano threefold: Namikawa proved that Gorenstein terminal Fano threefolds has a smoothing. Galkin developed the principal invariants to determine their associated Mori-Mukai family.

One of the main difficulties of the main problem for general terminal threefolds is the lack of classification of general terminal Fano varieties up to deformations.

Mitigation: Compute some sufficiently powerful invariants of X_2/Z_2.

Quantum invariants: Quantum period and toric Landau-Ginzburg models.

Sano: Fano threefolds with ordinary terminal singularities (i.e. singularities not of type cAx/4 in the classification) admits a $\mathbb{Q}$-Gorenstein deformation to terminal Fano threefolds with terminal quotient singularities.

In particular, we can define quantum invariants for them.

Definition 1.3 (Quantum periods) *Let $\mathfrak{X}$ be a smooth DM stack and X be its coarse moduli space, which is a $\mathbb{Q}$-factorial Fano threefold with quotient singularities. The quantum period of X is the power series*

$$G_X(t) = 1 + \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \tau_{-K_X\cdot\beta-2}\mathbf{1}\rangle_\beta \cdot t^{-K_X\cdot\beta}.$$

The regularized quantum period is the power series

$$\hat{G}_X(t) = 1 + \sum_{\beta \in H_2(X,\mathbb{Z})} (-K_X\cdot\beta)!\langle \tau_{-K_X\cdot\beta-2}\mathbf{1}\rangle_\beta \cdot t^{-K_X\cdot\beta}.$$

Definition 1.4 (toric Landau-Ginzburg models) *Let $\mathfrak{X}$ be a smooth DM stack and X be its coarse moduli space, which is a $\mathbb{Q}$-factorial Fano threefold with quotient singularities. A toric Landau-Ginzburg model of X is a Laurent polynomial f satisfying the following properties:*

1. *Period condition.* $\hat{P}_f(t) = \hat{G}_X(t)$.
2. *Calabi-Yau compactification. There exists a fiberwise compactification (the so called Calabi–Yau compactification) $Y \to \mathbb{C}$ such that Y is a smooth Calabi-Yau variety.*
3. *Polytope condition. There is a degeneration $X \rightsquigarrow X_T$ to a toric variety X_T whose fan polytope (the convex hull of generators of its rays) coincides with the Newton polytope (the convex hull of non-zero coefficients) of f.*

Quantum invariants are naturally invariant under $\mathbb{Q}$-Gorenstein deformations. They are expected to be "powerful enough" so that they are complete $\mathbb{Q}$-Gorenstein deformation invariants. We want to use it as a substitution of the classification: Given quantum invariants of a terminal Fano threefold X_1, determine the quantum invariants of all possible X_2/Z_2.

Overview

Main results

Obtain lower models by MMP

Construction 2.1 *Let X be a terminal Fano variety, $D \in \mathrm{Eff}_{\mathbb{R}}(X)$ an effective $\mathbb{R}$-divisor which is not big.*
Step 1: *Take a $\mathbb{Q}$-factorization and run a D-MMP to obtain a D-minimal model Y.*
Step 2: *D_Y is semi-ample and induces a morphism $Y \to Z$.*
Step 3: *Run $(-K_Y)$-MMP over Z to obtain a weak central model Y'/Z.*
Step 4: *$-K_{Y'}$ is semi-ample over Z and induces a morphism $Y' \to Z'/Z$. The fibration Y'/Z' is a central model.*

Every central models $X \geq Y'/Z'$ can be constructed as above for some D.

Definition 2.2 (Quantum periods for pairs) *Let $\mathfrak{X}$ be a smooth DM stack and X be its coarse moduli space, which is a $\mathbb{Q}$-factorial Fano threefold with quotient singularities. Let D be an $\mathbb{R}$-divisor on X. The quantum period of (X, D) is the power series*

$$G_{X,D}(t) = 1 + \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \tau_{-K_X\cdot\beta-2}\mathbf{1}\rangle_\beta \cdot e^{-D\cdot\beta} t^{-K_X\cdot\beta}.$$

The regularized quantum period is the power series

$$\hat{G}_{X,D}(t) = 1 + \sum_{\beta \in H_2(X,\mathbb{Z})} (-K_X\cdot\beta)!\langle \tau_{-K_X\cdot\beta-2}\mathbf{1}\rangle_\beta \cdot e^{-D\cdot\beta} t^{-K_X\cdot\beta}.$$

Definition 2.3 (toric Landau-Ginzburg models for pairs) *Let $\mathfrak{X}$ be a smooth DM stack and X be its coarse moduli space, which is a $\mathbb{Q}$-factorial Fano threefold with quotient singularities. A toric Landau-Ginzburg model of X is a Laurent polynomial f satisfying the following properties:*

1. *Period condition.* $\hat{P}_f(t) = \hat{G}_{X,D}(t)$.
2. *Calabi-Yau compactification. There exists a fiberwise compactification (the so called Calabi–Yau compactification) $Y \to \mathbb{C}$ such that Y is a smooth Calabi-Yau variety.*
3. *Polytope condition. There is a degeneration $X \rightsquigarrow X_T$ to a toric variety X_T whose fan polytope (the convex hull of generators of its rays) coincides with the Newton polytope (the convex hull of non-zero coefficients) of f.*

Definition 2.4 (Parametrized toric LG models) *Let $\mathfrak{X}$ be a smooth DM stack and X be its coarse moduli space, which is a $\mathbb{Q}$-factorial Fano threefold with quotient singularities. A parametrized toric Landau-Ginzburg model of X is a family of Laurent polynomials f over $\mathrm{Pic}_{\mathbb{R}}(X)$ such that f_D is a toric Landau-Ginzburg model of D.*

Theorem 2.5 (HS25) *Let X be a smooth Fano threefold.*

1. *Let $g : X \to Y$ be a divisorial contraction of a prime divisor E. Then we have*
$$\lim_{r\to+\infty} \hat{G}_{X,rE}(t) = \hat{G}_Y(t).$$
In particular, if X has a parametrized toric LG model and $\lim_{r\to+\infty} f_{rE}$ exists, then the limit is a toric LG model of Y.
2. *Let $h : X \to Z$ be a Mori fibre space with general fibre F. Let A be an ample divisor on Z. Then we have*
$$\lim_{r\to+\infty} \hat{G}_{X,rh^*A}(t) = \hat{G}_F(t).$$
In particular, if X has a parametrized toric LG model and $\lim_{r\to+\infty} f_{rh^*A}$ exists, then the limit reduces to a toric LG model of F.

Theorem 2.6 (HS, Ongoing) *Let $g : X \to Y$ be a divisorial contraction between $\mathbb{Q}$-factorial terminal Fano threefolds which contracts a prime divisor E to a point. Assume that X and Y have ordinary terminal singularities. Then we have*

$$\lim_{r \to +\infty} \hat{G}_{X,rE}(t) = \hat{G}_Y(t).$$

Sketch of the proof: For a threefold divisorial contraction $X \to Y$, we want to construct a family $\mathcal{W} \to C \ni 0$ such that:

1. The general fibre is isomorphic to Y.
2. The special fibre is a snc divisor containing X.

For smooth blow-ups the family is simply the degeneration to the normal cone. For a divisorial contraction to an ordinary double point the family is constructed in the study of conifold transitions. The construction for the general case heavily relies on the classification results.

For a threefold Mori fibration $X \to Z$, when X is smooth Fano, we have Z is a smooth Fano.

Overview

Some examples

Parametrized toric LG models of X can be computed in one of the following cases:

1. X degenerates to a Gorenstein toric Fano threefold.
2. X is a toric complete intersection of ample hypersurfaces.

Example 3.1 (Mori-Mukai No. 3.9) *Let X be the blow-up of $\mathbb{P}(1,1,1,2)$ at the vertex and a quartic curve C. The effective cone $\mathrm{Eff}_{\mathbb{R}}(X)$ is generated by 4 extremal rays $2H-2E_1-E_2, H-E_2, E_1, E_2$, where E_1 is the exceptional divisor centered at the vertex, E_2 is the exceptional divisor centered at C and H is the class of a hyperplane section. A toric degeneration is given by the rays* $(1,0,0),(0,1,0),(0,0,1),(-2,-1,-1),(-1,0,0),(-1,-1,-1),(0,-1,-$

Some examples

A parametrized toric LG model of X is given by

$$\tilde{f} = a_2 x + y + z + a_4 xy + a_4 xz + \frac{a_3}{x} + \frac{(a_4 x + 1)^2}{x^2 yz}$$

with the relation $(a_1, a_2, a_3, a_4) \sim (\lambda a_1, \lambda^{-2} a_2, \lambda^2 a_3, \lambda^{-1} a_4)$. On another chart, we can write the parametrized LG model as

$$\tilde{f} = a_2' x + a_1 y + a_1 z + a_4' xy + a_4' xz + \frac{1}{x} + \frac{(a_4' x + a_1)^2}{x^2 yz}.$$

The contraction in the direction of the rays of a_1 and a_4 gives a divisorial contraction to a $\frac{1}{2}(1,1,1)$ singularity with toric LG model

$$x + y + z + xy + xz + \frac{1}{yz} + \frac{2}{xyz} + \frac{1}{x^2 yz}.$$

The contraction in the direction of the rays of a_2 and a_3 gives Mori-Mukai 2.36.

Thank you.