Modular differential equations for weak D_n-Jacobi forms

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1. Jacobi modular forms: invariant theory, algebraic geometry, Kac-Moody algebras, ...

The Chevalley theorem for finite Coxeter groups: the invariant polynomials form a free graded algebra.

I.Bernstein and **O.Shvartsman** (1978): Complex crystallographic Coxeter groups. Theta-functions appeared: the Dynkin diagram and the parameter $\tau \in \mathbb{H}^+$.

M. Eichler, D. Zagier, The theory of Jacobi forms (1985);

K. Wirthmüller, Root systems and Jacobi forms (1992);

K. Saito, Extended Affine Root Systems II (1990);

B. Dubrovin, Geometry of 2D topological field theories in Integrable Systems and Quantum Groups (1996);

V. Gritsenko, Moduli space of polarized K3 surfaces and abelian surfaces (1994); Reflective modular forms and their applications (2010-2018);

V. Gritsenko, Jacobi modular forms: 30 ans après. Course at NRU HSE.

2. Jacobi forms in many abelian variables

See: V. Gritsenko, *Modular forms and moduli spaces of abelian and K3 surfaces.* Algebra i Analiz **6** (1994), 65–102.

Definition. A *weak Jacobi form* of *weight* k and *index* m with respect to an *even integral lattice* $L > 0$ is a holomorphic function $\varphi(\tau, \mathfrak{z})$ on $\mathbb{H}_1 \times (L \otimes \mathbb{C})$ which satisfies two equations

$$\varphi\Big(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\Big) = (c\tau + d)^k \exp\Big(\pi i m \frac{c(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\Big)\varphi(\tau, \mathfrak{z})$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and for $\forall\, \lambda, \mu \in L$

$$\varphi(\tau, \mathfrak{z} + \lambda\tau + \mu) = \exp\Big(-\pi i m\big((\lambda, \lambda)\tau + 2(\lambda, \mathfrak{z})\big)\Big)\varphi(\tau, \mathfrak{z}).$$

and has a Fourier expansion ($q = e^{2\pi i \tau}$, $\zeta^l = e^{2\pi i (\mathfrak{z}, l)}$, $l \in L^\vee$)

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \geqslant 0,\, l \in L^\vee} a(n, l) q^n \zeta^l,$$

The space $J^w_{k,m}(L)$ of all such Jacobi forms is finite dimensional.

3. Basic results: see Gritsenko-1994

1) $J^{w}_{k,m}(L) = J^{w}_{k,1}(L(m))$.

2) Let $\varphi_{k,1}(\tau,\mathfrak{z}) = \sum_{l\in L^\vee}\sum_{n\geqslant 0} a(n,l)q^n\zeta^l \in J^{w}_{k,1}(L)$.

The coefficient $a(n,l)$ depends only on $2n-(l,l)$ and $l \mod L$.

Moreover, $a(n,-l) = (-1)^k a(n,-l)$, and if $a(n,l)\neq 0$ then

$$2n-(l,l) \geqslant -\min_{u\in l+L}(u,u).$$

3) $f(\tau,z)\in J^{w}_{k,m}(L)$ is called holomorphic Jacobi form if $a(n,l)\neq 0$ implies $2nm-(l,l)\geq 0$. We have $J_{k,m}(L)\subset J^{w}_{k,m}(L)$.

4) If $J_{k,m}(L)\neq\{0\}$, then $k\geq \frac{\mathrm{rank}\ L}{2}$. The minimal possible (holomorphic) weight is called *singular*.

5) $J_{k,1}(L)$ is isomorphic to a space of vector-valued $SL_2(\mathbb{Z})$-modular forms of weight $k-\frac{\mathrm{rank}\ L}{2}$.

6) Lift: $J_{k,1}(L)\mapsto M_k(\widetilde{SO}^+(2U\oplus L(-1)))$ – the space of modular forms with respect to $\widetilde{SO}^+(2U\oplus L(-1)))$ in $n+2$ variables. We note that $\mathrm{sign}\,(2U\oplus L(-1)) = (2, n+2)$.

4. Polynomial rings of Jacobi forms

$\varphi(\tau, \mathfrak{z})$ is called $O(L)$- **or** $W(L)$-**invariant**, if $\varphi(\tau, w(\mathfrak{z})) = \varphi(\tau, \mathfrak{z})$ $\forall\, w \in O(L)$ or $w \in W(L)$.

Theorem. (Wirthmüller, 1992, affinisation of Chevalley theorem) *The ring*

$$J_{*,*}^{W(R)} = \bigoplus_{k\in\mathbb{Z},\, m\in\mathbb{Z}_{\geq 0}} J_{k,m}^{w,W(R)}(L)$$

is polynomial for any irreducible root system $R \neq E_8$, $R = L(R)$.

• In 2018, Haowu Wang showed that this fact is *not true* for E_8.

A constructive proof for E_6, A_n and G_2 was found in the papers of Ph.D. students of K. Saito and B. Dubrovin: I. Satake and M. Bertola (2000). We, with Dimitri Adler (2020, 2024), give a simple proof for D_n and C_n, then D. Adler for F_4.

5. The odd Jacobi theta-series $\vartheta(\tau, z)$

V. Gritsenko, V. Nikulin *Automorphic forms and Lorentzian Kac–Moody algebras. II.* Inter. J. Math. **9** (1998), 201–275.

$$\vartheta(\tau, z) = q^{\frac{1}{8}}(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1 - q^n\zeta)(1 - q^n\zeta^{-1})(1 - q^n) =$$

$$= \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(n+1)}{2}} \zeta^{n+\frac{1}{2}} \in J^{A_1, hol}_{1/2,\ 1/2}(v_\eta^3 \times v_H)$$

The functional equations for this theta-series:

$$\vartheta(\tau, z + x\tau + y) = (-1)^{x+y} e^{-\pi i(x^2\tau + 2xz)}\vartheta(\tau, z), \quad (x, y) \in \mathbb{Z}^2,$$

$$\vartheta(\tau, z) = v_\eta^3(M)(c\tau + d)^{\frac{1}{2}} e^{\pi i \frac{cz^2}{c\tau+d}}\vartheta(\tau, z), \quad M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$$

where v_η is a multiplier system of order 24 of the Dedekind η-function.

6. The Jacobi theta-series: special functions

The main special functions:

$$\frac{\partial\vartheta(\tau,z)}{\partial z}\Big|_{z=0} = 2\pi i\,\eta(\tau)^3 = 2\pi i\,\Big(q^{\frac{1}{24}}\prod_{n\ge 1}(1-q^n)\Big)^3,$$

$$\frac{\partial^2\log\vartheta(\tau,z)}{\partial z^2} = -\wp(\tau,z) - \frac{8}{24}\pi^2 E_2(\tau),$$

$$E_2(\tau) = \tfrac{24}{2\pi i}\tfrac{d\,\log(\eta(\tau))}{d\tau} = 1 - 24\sum_{n\ge 1}\sigma_1(n)q^n$$

The **Chazy equation** for the *quasi-modular* Eisenstein series $E_2(\tau)$:

$$y''' = 2yy'' - 3(y')^2.$$

7. The Jacobi theta-series: examples of Jacobi forms for A_1

The lattice of rank 1

$$A_1 = \langle 2 \rangle = \mathbb{Z}v, \quad v^2 = 2, \quad A_1^\vee = \mathbb{Z}\,(v/2).$$

$$\vartheta(\tau, z) \in J^{A_1}_{1/2,\ 1/2}(v_\eta^3 \times v_H).$$

$$\varphi^{A_1}_{-2,1}(\tau, z) = \vartheta(\tau, z)^2/\eta(\tau)^6 = (\zeta - 2 + \zeta^{-1}) + q(\ldots),$$

$$\varphi^{A_1}_{0,1}(\tau, z) = c\wp(\tau, z)\varphi^{A_1}_{-2,1}(\tau, z) = (\zeta + 10 + \zeta^{-1}) + q(\ldots)$$

are algebraically independent Jacobi forms of index 1,

$$J^{w, W(A_1)}_{*,*} = \bigoplus_{k\in\mathbb{Z},\, m\in\mathbb{Z}_{\geq 0}} J^{w}_{2k,m}(A_1) = M_*[\varphi_{-2,1}, \varphi_{0,1}] \quad ([\text{EZ } 1985]),$$

$$EG(K3; \tau, z) = 2\varphi_{0,1}(\tau, z),$$

$$\varphi^{A_1}_{-1,2}(\tau, z) = \vartheta(\tau, 2z)/\eta(\tau)^3, \quad \varphi^{A_1}_{0,\frac{3}{2}}(\tau, z) = \vartheta(\tau, 2z)/\vartheta(\tau, z).$$

8. Lattice D_n

The even positive definite quadratic lattice D_n

$$D_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid \sum x_i \equiv 0 \mod 2\}$$

is a sublattice of index 2 of the eucledien $\mathbb{Z}^n = \langle e_1, \dots, e_n \rangle_{\mathbb{Z}}$.
The root system $R(D_n)$ =the set of 2-vectors (roots) in D_n, $|R(D_n)| = 2n(n-1)$. $D_1 = \langle 4 \rangle$ has no 2-roots, $D_2 = A_1 \oplus A_1$.
Recall that

$$D_n^\vee / D_n = \{0, e_1, \frac{1}{2}(e_1 + \dots + e_{n-1} \pm e_n) \mod D_n\}.$$

The Weyl group $W(D_n)$ acts on the elements of the lattice D_n by permutations and changing of the sign of even number of coordinates $(x_1, \dots, x_n)$. If $n \neq 4$ then

$$O(D_n) = W(C_n), \quad O(D_n)/W(D_n) = S_2,$$

$$O(D_4) = W(F_4), \quad O(D_4)/W(D_4) = S_3.$$

9. Invariant theory and the ring of weak Jacobi forms

Theorem. *For any $n \geq 2$*

$$J^{w,O(D_n)}_{*,*} = M_*[\,\varphi^{D_n}_{-4,1}, \varphi^{D_n}_{-2,1}, \varphi^{D_n}_{0,1};\ \varphi^{D_n}_{-6,2}, \ldots, \varphi^{D_n}_{-2n+2,2}, (\omega^{D_n}_{-n,1})^2\,].$$

Here one puts $W(C_n)$ instead of $O(D_n)$ for $n = 4$.

About the proof [AG 2020, 2024].

1) We note that $D_n < \mathbb{Z}^n$, and $D_n(2) < \mathbb{Z}(2)^{\oplus n} = nA_1$.
The Jacobi forms of index 1 for the lattice nA_1 are Jacobi forms of index 2 for the lattice D_n! We can construct weak D_n-Jacobi forms of index 2 using the generators of index 1 for A_1.

2) $\varphi(\tau, z_1, \ldots, z_n)|_{z_n=0} \in J^{w}_{*,*}(D_{n-1})$. We can use induction by n.

3) $J^{w,W(D_2)}_{*,*}$ is not free! But this is true for $O(D_2)$-forms.

4) **How to construct the non-trivial generators** $\varphi^{D_n}_{-4,1}$, $\varphi^{D_n}_{-2,1}$ and $\varphi^{D_n}_{0,1}$ for $n > 2$? If the theorem is true, then the generators $\varphi^{D_n}_{-4,1}$, $\varphi^{D_n}_{-2,1}$ are determined uniquely, and $\varphi^{D_n}_{0,1}$ belongs to a two dimensional space.

10. Jacobi theta-series: examples for D_n

$$\vartheta(\tau, z) = v_\eta^3(A)(c\tau + d)^{\frac{1}{2}} e^{\pi i \frac{cz^2}{c\tau+d}} \vartheta(\tau, z).$$

$$\vartheta(\tau, z + x\tau + y) = (-1)^{x+y} e^{-\pi i(x^2\tau + 2xz)} \vartheta(\tau, z), \quad (x, y) \in \mathbb{Z}^2.$$

Taking the "direct" product we get

$$\Theta_{D_8} = \vartheta(\tau, z_1) \cdot \ldots \vartheta(\tau, z_8) \in J_{4,1}(D_8) \quad \text{(singular weight!)}$$

$$\psi_{5,1}^{D_7} = \eta^3(\tau)\vartheta(\tau, z_1) \cdot \ldots \cdot \vartheta(\tau, z_7) \in J_{5,1}^{cusp}(D_7),$$

$$\omega_{-n,1}^{D_n} = \frac{\vartheta(\tau, z_1)}{\eta(\tau)^3} \cdot \ldots \cdot \frac{\vartheta(\tau, z_n)}{\eta(\tau)^3} \in J_{-n,1}^{w,W(D_n)},$$

For different method for construction of Jacobi forms for root systems see F. Cléry, V. Gritsenko, *Modular forms of orthogonal type and Jacobi theta-series.* Abh. Math. Semin. Univ. Hambg. **83** (2013), 187–217.

11. The first non-trivial D_n-generator

[GN-1996/98]: We constructed $\varphi_{0,1}^{A_1}$ using $\varphi_{-2,1}^{A_1}$:

$$\varphi_{0,1}^{A_1}(\tau,z) = 2\frac{\varphi_{-2,1}^{A_1}|T_-(2)}{\varphi_{-2,1}^{A_1}(\tau,z)} = \frac{1}{2^2}\cdot\frac{\varphi_{-2,1}^{A_1}(2\tau,2z)}{\varphi_{-2,1}^{A_1}(\tau,z)} + \frac{\varphi_{-2,1}^{A_1}(\frac{\tau}{2},z)}{\varphi_{-2,1}^{A_1}(\tau,z)} + \frac{\varphi_{-2,1}^{A_1}(\frac{\tau+1}{2},z)}{\varphi_{-2,1}^{A_1}(\tau,z)}.$$

Similar to this, we can construct a Jacobi form of weight 0 for D_n

$$\varphi_{0,1}^{D_n} = 2^{n+1}\frac{\omega_{-n,1}^{D_n}|T_-(2)}{\omega_{-n,1}^{D_n}} = \frac{\omega_{-n,1}^{D_n}(2\tau,2\mathfrak{z})}{\omega_{-n,1}^{D_n}(\tau,\mathfrak{z})} + 2^n\frac{\omega_{-n,1}^{D_n}(\frac{\tau}{2},\mathfrak{z})}{\omega_{-n,1}^{D_n}(\tau,\mathfrak{z})} + 2^n\frac{\omega_{-n,1}^{D_n}(\frac{\tau+1}{2},\mathfrak{z})}{\omega_{-n,1}^{D_n}(\tau,\mathfrak{z})} =$$

$$= \frac{\vartheta_2(\tau,z_1)\ldots\vartheta_2(\tau,z_n)}{2\vartheta_2^n(\tau,0)} + \frac{\vartheta_3(\tau,z_1)\ldots\vartheta_3(\tau,z_n)}{2\vartheta_3^n(\tau,0)} + \frac{\vartheta_4(\tau,z_1)\ldots\vartheta_4(\tau,z_n)}{2\vartheta_4^n(\tau,0)}.$$

MDO (modular differential operator) $H_k : J_{k,m}^w \to J_{k+2,m}^w$

$H_{-4}(\varphi_{-4,1}^{D_n}) = c\varphi_{-2,1}^{D_n}$ and $H_{-2}(\varphi_{-2,1}^{D_n}) \in J_{0,1}^{W(C_n)}$.

We proposed some "**integration**", i.e. an opposite operation!

12. Modular differential operator $\mathbb{D}_k$ on $M_k(SL_2(\mathbb{Z}))$

We put $\mathbb{D} = 12q\frac{d}{dq} = 6\pi i\frac{d}{d\tau}$. For any automorphic function $f(\tau)$ of weight 0 we get a form of weight 2: $\mathbb{D}(f) \in M_2^{(mer)}(SL_2(\mathbb{Z}))$.
Therefore,

$$\mathbb{D}_k : M_k(SL_2(\mathbb{Z})) \to M_{k+2}(SL_2(\mathbb{Z})), \quad \mathbb{D}_k(f) = 12D(f) - kE_2 \cdot f,$$

where $E_2(\tau) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$.

Ramanujan's system for generators of $M_*^{quasi} = \mathbb{C}[E_2, E_4, E_6]$

$$\mathbb{D}_2(E_2) = -E_2^2 - E_4, \ \ \mathbb{D}_4(E_4) = -4E_6, \ \mathbb{D}_6(E_6) = -6E_4^2.$$

The **Chazy equation** for the quasi-modular Eisenstein series $E_2(\tau)$:
$y''' = 2yy'' - 3(y')^2$.

Kaneko-Zagier equation (for characters of vertex algebras):

$$f''(\tau) - \tfrac{k+1}{6}E_2(\tau)f'(\tau) + \tfrac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0,$$

$$\mathbb{D}_{k+2} \circ \mathbb{D}_k(f) - k(k+2)E_4 \cdot f = 0.$$

13. MDO: Modular differential operator for Jacobi forms

Let $L = \langle e_1, \dots, e_n \rangle_{\mathbb{Z}}$ be a lattice with the inner product $(\cdot,\cdot) > 0$.
Let e_1^*, e_2^*, ..., e_n^* be the dual basis of $L^\vee$. For
$\mathfrak{z} = z_1 e_1 + \dots + z_n e_n \in L \otimes \mathbb{C}$ we define $\frac{\partial}{\partial \mathfrak{z}} = \sum_{i=1}^n e_i^* \frac{\partial}{\partial z_i}$.
Let $\varphi_{k,m}(\tau, \mathfrak{z})$ be a Jacobi form of weight k and index m for the lattice L. We define

$$H_k^L(\varphi_{k,m}(\tau,\mathfrak{z})) =$$

$$= \left(\mathbb{D}\varphi_{k,m} + \frac{3}{2\pi^2 m} \left(\frac{\partial}{\partial \mathfrak{z}}, \frac{\partial}{\partial \mathfrak{z}} \right) \varphi_{k,m} \right) + \left(\frac{\operatorname{rk} L}{2} - k\right) E_2(\tau) \varphi_{k,m}$$

is a Jacobi form of weight $k+2$ and the same index m. In the brackets, we put the heat operator H^L for the lattice L. For weight $k = \frac{\operatorname{rk} L}{2}$ we have $H_k^L = H^L$. In particular,

$$H^{A_1}\big(\vartheta(\tau,z)\big) = 0, \quad H^{D_8}\big((\vartheta(\tau,z_1)\cdot\vartheta(\tau,z_2)\cdot\ldots\cdot\vartheta(\tau,z_8)\big) = 0.$$

14. MI: Modular "integration" for Jacobi forms

$$H_k(\varphi_{k,m})(\tau,\mathfrak{z}) =$$

$$= \sum_{n=0}^{\infty}\sum_{l\in L^\vee}\left(12n - \frac{6}{m}(l,l)\right) a(n,l)q^n\zeta^l + \left(\frac{\operatorname{rk} L}{2} - k\right) E_2(\tau)\varphi_{k,m}(\tau,\mathfrak{z}).$$

We find a combination of MDO of $\varphi_{0,1}^{D_n}$ without q^0-term in its Fourier expansion. Then we divide it by $\Delta(\tau)$

$$\varphi_{-4,1}^{D_n} = \frac{2E_4H_2H_0(\varphi_{0,1}^{D_n}) + (n+4)E_6H_0(\varphi_{0,1}^{D_n}) - n^2E_4^2\varphi_{0,1}^{D_n}}{576n(n+4)\Delta},$$

$$\varphi_{-2,1}^{D_n} = \frac{2E_6H_2H_0(\varphi_{0,1}^{D_n}) + (n+4)E_4^2H_0(\varphi_{0,1}^{D_n}) - n^2E_4E_6\varphi_{0,1}^{D_n}}{1152n(n+4)\Delta}.$$

15. The proof of Wirthmüller theorem for $O(D_n) = W(C_n)$

D. Adler, V. Gritsenko: *Modular differential equations of $W(D_n)$-invariant Jacobi forms* (2024).

$$
\begin{array}{cccccccc}
\varphi^{D_n}_{0,1} & \varphi^{D_n}_{-2,1} & \varphi^{D_n}_{-4,1} & \varphi^{D_n}_{-6,2} & \cdots & \cdots & \varphi^{D_n}_{-2n+2,2} & (\omega^{D_n}_{-n,1})^2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi^{D_{n-1}}_{0,1} & \varphi^{D_{n-1}}_{-2,1} & \varphi^{D_{n-1}}_{-4,1} & \varphi^{D_{n-1}}_{-6,2} & \cdots & \varphi^{D_{n-1}}_{-2n+4,2} & (\omega^{D_{n-1}}_{-n+1,1})^2 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\varphi^{D_4}_{0,1} & \varphi^{D_4}_{-2,1} & \varphi^{D_4}_{-4,1} & \varphi^{D_4}_{-6,2} & (\omega^{D_4}_{-4,1})^2 & 0 & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\
\varphi^{D_3}_{0,1} & \varphi^{D_3}_{-2,1} & \varphi^{D_3}_{-4,1} & (\omega^{D_3}_{-3,1})^2 & 0 & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & & & & \\
\hat{\varphi}^{D_2}_{0,1} & \varphi^{D_2}_{-2,1} & \varphi^{D_2}_{-4,1} & 0 & & & &
\end{array}
$$

Here the Jacobi forms from the last line are suitable modifications of $\varphi^{A_1}_{-2,1}$ and $\varphi^{A_1}_{0,1}$. All vertical arrows are the restrictions from D_n to D_{n-1} by vanishing the last coordinate z_n together with multiplication of the restriction by a suitable non-zero constant number.

16. Analog of the Ramanujan system for D_n

Theorem ([AG2024]). The following system is true for any $n \geq 2$

$$\begin{cases} H_{-4}(\varphi^{D_n}_{-4,1}) - (n-4)\varphi^{D_n}_{-2,1} = 0 \\ 4H_{-2}(\varphi^{D_n}_{-2,1}) - 3n\varphi^{D_n}_{0,1} - (n-8)E_4(\tau)\varphi^{D_n}_{-4,1} = 0 \\ 3H_0(\varphi^{D_n}_{0,1}) - 2nE_4(\tau)\varphi^{D_n}_{-2,1} - nE_6(\tau)\varphi^{D_n}_{-4,1} = 0 \end{cases}$$

We give also individual MDEs for all generators. We see that there are special simplifications for $n = 4, 8$. Moreover, there is also an anomaly for $n = 12$!

“Multiparametric” elliptic genera are related to Jacobi forms from this bigraded ring. In $N = 2$ SCFT elliptic genera are generalisations of chiral characters (Ch. Keller). One can calculate elliptic genera in terms of Gromov-Witten invariants in a dual string compactification (G. Oberdieck, A. Pixton, ...).

17. MDEs: review of our results for A_1, $D_2{=}A_1 \oplus A_1$, D_n

- The simplest MDEs of **degree 1** *with respect to the heat operator*. (The theta-functions are in the kernel of the corresponding heat operator.)

Our list of the **"theta"-generators** is the following:

$$EG(CY_3),\ \varphi_{-4,1}^{D_4} \text{ and } \textit{only one Jacobi form } \psi_{0,1}^{D_{12}} \textit{ in all } J_{0,1}^{O(D_n)},$$

where $\psi_{0,1}^{D_n} = \varphi_{0,1}^{D_n} - E_4\varphi_{-4,1}^{D_n}$.

The Jacobi form $\psi_{0,1}^{D_{12}}$ is an almost full analog of $EG(CY_3)$ from the point of view of automorphic Borcherds products!

- MDE of **degree 2**. *A generalisation of Kaneko-Zagier's equation for characters of vertex algebras.* Our list:

$$\varphi_{-2,1}^{D_4} \text{ and the Jacobi form } \psi_{0,1}^{D_8} \in J_{0,1}^{O(D_8)}.$$

$\psi_{0,1}^{D_8}$ is the partition functions for the Borcherds-Enriques Kac-Moody algebra.

Jacobi forms of weigh -2 should be related to Gromov-Witten invariants.

18. MDEs: review

- MDE of **degree 3** . Our list:

  $EG(K3)$, $EG(CY_5)$ and $EG(CY_4)$ with $e(CY_4) = 48, -18$;

  all generators $\varphi^{D_n}_{-4,1}$ $(n \neq 4)$;

  only three special Jacobi forms in $J_{0,1}^{O(C_n)}$:

$$\varphi^{D_n}_{0,1}\ ,\quad \psi^{D_n}_{0,1} = \varphi^{D_n}_{0,1} - E_4\varphi^{D_n}_{-4,1}\ ,\quad \rho^{D_n}_{0,1} = (n-4)\varphi^{D_n}_{0,1} - 8E_4\varphi^{D_n}_{-4,1};$$

  all Jacobi forms in the two dimensional space $J_{0,1}^{W(D_4)}$.

- MDE of **degree 4**. All generators $\varphi^{D_n}_{-2,1}$ for $n \neq 4$:

$$4H^{[4]}_{-2}(\varphi^{D_n}_{-2,1}) - (3n^2 - 12n + 224)E_4H^{[2]}_{-2}(\varphi^{D_n}_{-2,1}) +$$

$$(n^3+24n^2-144n+384)E_6H_{-2}(\varphi^{D_n}_{-2,1}) - 12(n-8)(n-2)(n+4)E_4^2\varphi^{D_n}_{-2,1} = 0.$$

- MDE of **deg**$=$ **5**.

$EG(Hilb^2(K3))$, $EG(Kum^2(A))$, a **generic** Jacobi form in $J_{0,1}^{W(D_n)}$.