Decomposition of Lie algebra into sum of two subalgebras and Integrability

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Introduction

Let $\mathfrak{g}$ be a Lie algebra with a basis $\mathbf{e}_i$, $i = 1, \ldots, n$. Suppose we have a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \tag{1}$$

where $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are subalgebras in $\mathfrak{g}$. The simplest example is the Gauss decomposition of the matrix algebra into the sum of upper and law triangular matrices.

Let us consider the following non-linear system of ODEs

$$\frac{dU}{dt} = [\pi_+(U), U], \qquad U(0) = U_0. \tag{2}$$

Here

$$U(t) = \sum_1^n u_i(t)\, \mathbf{e}_i,$$

and π_+ denotes the projector onto $\mathfrak{g}_+$ parallel to $\mathfrak{g}_-$. Very often we denote by X_+ and X_- the projections of the element X on $\mathfrak{g}_+$ and on $\mathfrak{g}_-$, respectively. For simplicity, we assume that the algebra $\mathfrak{g}$ is embedded in a matrix algebra.

Proposition (Adler-Kostant-Symes scheme). The solution of the Cauchy problem (2) is given by the formula

$$U(t) = A(t)\, U_0\, A^{-1}(t), \tag{3}$$

where the function $A(t)$ is defined as the solution of the following factorization problem:

$$A^{-1}\, B = \exp\,(-U_0\, t), \qquad A \in G_+, \quad B \in G_-. \tag{4}$$

Here G_+ and G_- are the Lie groups of the algebras $\mathfrak{g}_+$ and g_-, respectively.

Proof. Differentiating (3) by t, we get

$$U_t = A_t\, U_0\, A^{-1} - A\, U_0\, A^{-1} A_t\, A^{-1} = [A_t\, A^{-1},\, U].$$

From (4) it follows that

$$-A^{-1} A_t A^{-1} B + A^{-1} B_t = -U_0 A^{-1} B.$$

The latter relation is equivalent to the equality

$$-A_t A^{-1} + B_t B^{-1} = -A U_0 A^{-1}.$$

Projecting it onto $\mathfrak{g}_+$, we get $A_t A^{-1} = U_+$, which proves the equality (2).

Integrals of motion, symmetries and Lie theorem

We associate with any dynamical system

$$\frac{d\,u_i}{dt} = F_i(u_1,\ldots,u_n), \qquad i = 1,\ldots,n\,. \tag{5}$$

the vector field

$$X_F = \sum_{k=1}^{n} F_k \frac{\partial}{\partial u_k}. \tag{6}$$

It is clear that function $J(u_1,\ldots,u_n)$ is a first integral iff

$$X_F(J) = 0.$$

Thus, first integrals of a dynamical system can be defined as elements of the kernel space for the corresponding vector field X_F.

Any function of first integrals is a first integral. Only functionally independent first integrals it is important to count.

Definition. First integrals $\phi_k(u_1, \dots, u_n), \quad k = 1, \dots, m$ are called functionally independent if the Jacobi matrix

$$\frac{D(\phi_1, \dots, \phi_m)}{D(u_1, \cdots, u_n)} = \begin{vmatrix} \frac{\partial \phi_1}{\partial u_1} & \dots & \frac{\partial \phi_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial \phi_m}{\partial u_1} & \dots & \frac{\partial \phi_m}{\partial u_n} \end{vmatrix}$$

has maximal rank.

Symmetries

The next fundamental concept of the local theory of nonlinear ODEs is the infinitesimal symmetry.

Definition. A vector field

$$X_G = \sum_{k=1}^{n} G_k(u_1, u_2, \dots, u_n) \frac{\partial}{\partial u_k},$$

is called (infinitesimal) symmetry of dynamical system (5) iff

$$[X_F, X_G] = 0. \tag{7}$$

Lie's Theorem.

Both first integrals and symmetries are very useful if we want to integrate dynamical system (5) by quadratures.

Suppose we know $n-1$ functionally independent first integrals $I_1, \ldots, I_{n-1}$ of (5). Making a change of variables

$$\hat{u}_1 = \phi_1(u_1, \ldots, u_n)\,, \hat{u}_2 = I_1(u_1, \ldots, u_n)\,, \ldots, \hat{u}_n = I_{n-1}(u_1, \ldots, u_n)\,,$$

for some ϕ_1, we get a system of the form

$$\frac{d\hat{u}_1}{dt} = \hat{f}_1(\hat{u}_1, \ldots, \hat{u}_n), \qquad \frac{d\hat{u}_2}{dt} = 0, \ \ldots, \ \frac{d\hat{u}_n}{dt} = 0,$$

which can be easily integrated in quadratures.

The procedure of integrating (5) if $n-1$ symmetries

$$X_1 = \sum_{k=1}^{n} G_k^1 \frac{\partial}{\partial u_k}, \quad X_2 = \sum_{k=1}^{n} G_k^2 \frac{\partial}{\partial u_k}, \ldots, \quad X_{n-1} = \sum_{k=1}^{n} G_k^{n-1} \frac{\partial}{\partial u_k} \tag{8}$$

are given, is not so standard.

For an efficient use of symmetries (8) we have to impose some restrictions on the structure of the Lie algebra generated by the vector fields X_i. The simplest version of a statement of such a sort reads as follows.

Theorem. Suppose dynamical system (5) has $(n-1)$ symmetries (8) such that

- the matrix
$$\begin{pmatrix} F_1 & F_2 & \dots & F_n \\ G_1^1 & G_2^1 & \dots & G_n^1 \\ \dots & \dots & \dots & \dots \\ G_1^{n-1} & G_2^{n-1} & \dots & G_n^{n-1} \end{pmatrix} \tag{9}$$
is non-degenerate;
- and
$$[X_i, X_j] = 0, \qquad 1 \leqslant i, j \leqslant n-1.$$

Then (5) can be integrated in quadratures.

Proof. It turns out that we can explicitly find a transformation σ of the form

$$\hat{u}_1 = \phi_1(u_1, \ldots, u_n)\,, \ldots, \hat{u}_n = \phi_n(u_1, \ldots, u_n)\,. \tag{10}$$

such that

$$\sigma(X_F) = \frac{\partial}{\partial \hat{u}_1}, \qquad \sigma(X_i) = \frac{\partial}{\partial \hat{u}_i}, \quad i = 1, \ldots, n-1.$$

Indeed, the unknown functions $\phi_i(u_1, \ldots, u_n)$ must satisfy the following system of equations

$$X_F(\phi_1) = 1, \qquad X_F(\phi_i) = 0, \quad i > 1$$

$$X_i(\phi_j) = \delta^i_j.$$

In particular, the function ϕ_1 satisfies the following conditions

$$X_F(\phi_1) = 1, \quad X_1(\phi_1) = 0, \quad \ldots, X_{n-1}(\phi_1) = 0.$$

Let us consider these relations as system of algebraic linear equations with respect to unknowns $z_i = \frac{\partial \phi_1}{\partial u_i}$. Since the determinant of matrix (9) is not zero, the system has an unique solution. It follows from the Frobenious Theorem that $\frac{\partial z_j}{\partial u_i} = \frac{\partial z_i}{\partial u_j}$.

Now, to reconstruct the function ϕ_1 one has to perform a sequence of integrations with respect to variables $u_1, u_2, \ldots, u_n$. In the similar way we can find the functions $\phi_2, \ldots, \phi_n$.

More general statement which involves both integrals and symmetries can be formulated as follows

Theorem. Suppose dynamical system (5) has k symmetries of the form (8) and (n-k-1) functionally independent first integrals $I_1, \ldots, I_{n-k-1}$ such that

- the matrix
$$\begin{pmatrix} F_1 & F_2 & \ldots & F_n \\ G_1^1 & G_2^1 & \ldots & G_n^1 \\ \ldots & \ldots & \ldots & \ldots \\ G_1^k & G_2^k & \ldots & G_n^k \end{pmatrix}$$
has the maximal rank;

- $$[X_i,\, X_j] = 0, \qquad 1 \leqslant i, j \leqslant n-1.$$

- and
$$X_i(I_j) = 0, \qquad 1 \leqslant i \leqslant k, \quad 1 \leqslant j \leqslant n-k-1.$$

Then (5) can be integrated in quadratures.

Hamiltonian properties

The Hamiltonian structure establishes relations between first integrals and symmetries.

Suppose we have a manifold with coordinates $y_1, \ldots, y_m$. Any Poisson bracket between functions $f(y_1, \ldots, y_m)$ and $g(y_1, \ldots, y_m)$ is given by the formula

$$\{f,\, g\} = \sum_{i,j} P_{i,j}(y_1, \ldots, y_m) \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j}, \tag{11}$$

where $P_{i,j} = \{y_i, y_j\}$. The functions P_{ij} are not arbitrary, since by definition we must have

$$\{f, g\} = -\{g, f\},$$

$$\{\{f, g\}, h\} + \{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

The algebra of all polynomials in the variables y_i, endowed by the operation $\{\cdot, \cdot\}$, is called *Poisson algebra*.

Formula (11) can be rewritten as

$$\{f,\, g\} = \langle \text{grad } f,\, \mathcal{H}(\text{grad } g)\rangle, \tag{12}$$

where $\mathcal{H} = \{P_{i,j}\}$ and $\langle\cdot,\cdot\rangle$ is a standard scalar product. The matrix $\mathcal{H}$ is called *Hamiltonian operator* or *Poisson tensor*.

The Poisson bracket is called *degenerate* if $\operatorname{Det}\mathcal{H} = 0$.

The Hamiltonian dynamical system are defined by the formula

$$\frac{dy_i}{dt} = \{H, y_i\}, \qquad i = 1, \ldots, m,$$

where H is a function of Hamilton.

Example. Consider a symplectic manifold with coordinates q_i and p_i, $i = 1, \dots n$. The standard (non-degenerate) Poisson constant brackets are given by formulas

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \qquad \{p_i, q_j\} = \delta_{i,j}, \tag{13}$$

where δ is the Kronecker symbol. Written in the variables

$$y_1 = p_1, \dots, y_n = p_n, \quad y_{n+1} = q_1, \dots, y_{2n} = q_n$$

the Poisson tensor $\mathcal{H}$ for the brackets (13) has the following block structure

$$\mathcal{H} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

In the case of *linear Poisson brackets* we have

$$P_{ij} = \sum_k b_{ij}^k y_k, \qquad i, j, k = 1, \dots, m.$$

It is well known that the formula (11) defines a Poisson bracket iff b_{ij}^k are *structural constants of some Lie algebra*. Very often, the name of this Lie algebra is assigned to the corresponding Poisson bracket.

Hamiltonian structure for equation (2)

Suppose that a Lie algebra $\mathfrak{g}$ is represented as

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-;$$

then the formula

$$[x, y]_R = 1/2\,([Rx, y] + [x, Ry]) \tag{14}$$

defines a second structure of Lie algebra on the vector space $\mathfrak{g}$.

Here $R = \pi_+ - \pi_-$ is the difference of projectors on $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively. If $x, y \in \mathfrak{g}$ are represented as $x = x_+ + x_-$ and $y = y_+ + y_-$ then the new Lie structure corresponds to the direct sum of ideals:

$$[x, y]_R = [x_+, y_+] - [x_-, y_-].$$

The operator R is the simplest example of the so called R-matrix. In general, the R-matrix is an linear operator $R : \mathfrak{g} \mapsto \mathfrak{g}$ that satisfy the modified Yang–Baxter equation

$$R\Big([y, R(x)] - [x, R(y)]\Big) + [R(x), R(y)] + [x, y] = 0,$$

where $x, y \in \mathfrak{g}$.

Thus, we have two linear Poisson brackets $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}_R$ on the symmetric algebra $S(\mathfrak{g})$, corresponding the commutators $[\cdot, \cdot]$ and $[\cdot, \cdot]_R$.

Lemma. Suppose a matrix U satisfies equation

$$\frac{dU}{dt} = [A, U], \tag{15}$$

then

i) if U_1 and U_2 satisfy (30), then the product $\bar{U} = U_1 U_2$ also satisfies (30);

ii) $\bar{U} = U^m$ satisfies (30) for all $m \in \mathbb{N}$;

iii) $\operatorname{tr} U^n$ is an integral of motion for (30);

Proof. Item i). We have

$$\bar{U}_t = (U_1)_t U_2 + U_1 (U_2)_t = [A, U_1] U_2 + U_1 [A, U_2] = A\,\bar{U} - \bar{U}\,A.$$

Item (ii) follows from (i). Item iii): Applying the trace functional to both sides of the identity $(U^n)_t = [A, U^n]$, we obtain $(\operatorname{tr} U^n)_t = 0$.

It follows from the lemma that $\operatorname{tr} U^n$ are first integrals of (2). The same fact follows also from formula (3). Moreover, (3) implies that any invariant of action of G_+ on $\mathfrak{g}$ is a first integral of (2). In spite of (3) we have

$$U(t) = B(t)\, U_0\, B^{-1}(t), \tag{16}$$

and therefore the invariants of G_--action are also integrals of motions.

In particular, consider the decomposition

$$\mathfrak{gl}_n = \mathfrak{n}_+ \oplus \mathfrak{b}_- \tag{17}$$

where the nilpotent subalgebra $\mathfrak{n}_+$ spanes by e_{ij} for $i < j$, and the Borel subalgebra $\mathfrak{b}_-$ is generated by e_{ij} for $i \geq j$. A problem that arises here is to describe the invariants of action

$$U \to BUB^{-1},$$

where $U \in \mathfrak{gl}_n$ and B is a low-triangular nondegenerate matrix.

Example. Let $n = 4$. The function

$$I = u_{22} + u_{33} - \frac{u_{12}u_{24} + u_{13}u_{34}}{u_{14}}$$

is an invariant.

Lemma. Equation (2) is Hamiltonian with the Poisson bracket $\{\cdot, \cdot\}_R$, where $R = \pi_+ - \pi_-$, and Hamiltonian $H = \operatorname{trace} U^2$.

Remark. Generally speaking the invariants of action G_+ and G_- do not commute with each other.

Reductions

From formula (3) it follows that if the initial data U_0 for the system (2) belongs to some $\mathcal{G}_+$-module $\mathcal{M}$, then $X(t) \in \mathcal{M}$ for any t. This specialization of equation (2) can be written as

$$M_t = [\pi_+(M),\, M], \qquad M \in \mathcal{M}. \tag{18}$$

Example. In the case of the decomposition $\mathfrak{gl}_n = \mathfrak{so} \oplus \mathfrak{b}_-$ we can take the vector space of symmetric matrices for $\mathcal{M}$. The corresponding system (18) is called *full symmetric Toda lattice*.

Introducing the product

$$M_1 \circ M_2 = [\pi_+(M_1),\, M_2], \qquad M_i \in \mathcal{M}, \tag{19}$$

we endow $\mathcal{M}$ with a structure of (generally speaking, non-commutative and non-associative) algebra. The system (18) is called $\mathcal{M}$-*reduction*, and the operation (19) is called $\mathcal{M}$-*product*.

Some classes of modules $\mathcal{M}$ correspond to interesting non-associative algebras defined by the formula (19).

We will use the following notation:

$$\mathrm{As}(X,Y,Z) = (X \circ Y) \circ Z - X \circ (Y \circ Z), \tag{20}$$

$$[X,Y,Z] = \mathrm{As}(X,Y,Z) - \mathrm{As}(Y,X,Z). \tag{21}$$

Definition. Algebras defined by the identity $[X,Y,Z] = 0$ are called *left-symmetric*

Definition. An algebra with identities

$$[X,Y,Z] + [Y,Z,X] + [Z,X,Y] = 0, \tag{22}$$

and

$$V \circ [X,Y,Z] = [V \circ X, Y, Z] + [X, V \circ Y, Z] + [X, Y, V \circ Z] \tag{23}$$

is called a *G-algebra*.

Remark. The identity (22) means that the operation $X \circ Y - Y \circ X$ is a Lie bracket.

Reductions in the case of $\mathbb{Z}_2$-graded Lie algebras

Let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \qquad (24)$$

be a $\mathbb{Z}_2$-graded Lie algebra, i.e.

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \qquad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \qquad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0.$$

Suppose that we have a decomposition (1), where $\mathfrak{g}_+ = \mathfrak{g}_0$. Consider $\mathfrak{g}_1$-reductions.

It is clear that

$$\mathfrak{g}_- = \{m - R(m) \,|\, m \in \mathfrak{g}_1\}, \qquad (25)$$

where $R = \pi_+$ is the projection onto $\mathfrak{g}_+ = \mathfrak{g}_0$ parallel to $\mathfrak{g}_-$.

The vector space (25) is a Lie subalgebra in $\mathfrak{g}$ iff the operator $R : \mathfrak{g}_1 \to \mathfrak{g}_0$ satisfies the modified Yang–Baxter equation

$$R\Big([R(x),\, y] - [R(y),\, x]\Big) - [R(x),\, R(y)] - [x,\, y] = 0,$$

where $x, y \in \mathfrak{g}_1$.

Remark. It is important to note that in our case R is an operator defined on $\mathfrak{g}_1$ and acting from $\mathfrak{g}_1$ in $\mathfrak{g}_0$, while as usual R is assumed to be an operator on $\mathfrak{g}$.

Proposition 1. If $[\mathfrak{g}_1,\, \mathfrak{g}_1] = \{0\}$, then $\mathfrak{g}_1$ is a left-symmetric algebra with respect to product (19).

Without the assumption $[\mathfrak{g}_1,\, \mathfrak{g}_1] = \{0\}$ we arrive at G-algebras:

Proposition 2.

i) The vector space $\mathfrak{g}_1$ is a G-algebra with respect to the operation (19).

ii) Any G-algebra can be obtained from an appropriate $\mathbb{Z}_2$-graded Lie algebra using this construction.

Decomposition of loop algebas

Let $\mathfrak{g}$ be a Lie algebra with a basis $\mathbf{e}_i$, $i = 1, \dots, n$. The Lie algebra $\mathfrak{g}((\lambda))$ of formal series of the form

$$\sum_{i=-n}^{\infty} g_i \lambda^i \quad | \quad g_i \in \mathfrak{g}, \quad n \in \mathbb{Z} \tag{26}$$

is called the (extended) *loop algebra* over $\mathfrak{g}$.

Consider decompositions

$$\mathfrak{g}((\lambda)) = \mathfrak{g}[[\lambda]] \oplus \mathcal{U} \tag{27}$$

of the loop algebra into a direct sum of vector subspaces, the first of which is the Lie subalgebra $\mathfrak{g}[[\lambda]]$ of all Taylor series, and the second one is a Lie subalgebra. The Lie algebra $\mathcal{U}$ is called *factoring*, or *complementary*.

The simplest factoring subalgebra consists of polynomials in $\frac{1}{\lambda}$ with a zero free term:

$$\mathcal{U}^{st} = \left\{ \sum_{i=1}^{n} g_i \lambda^{-i} \quad | \quad g_i \in \mathfrak{g}, \quad n \in \mathbb{N} \right\}. \tag{28}$$

Example 1. Let $\mathfrak{g} = \mathfrak{so}_3$ with the basis

$$\mathbf{e_1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{e_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then the elements

$$\mathbf{E}_{1,1} = \frac{\sqrt{1 - p\lambda^2}}{\lambda}\,\mathbf{e}_1, \qquad \mathbf{E}_{2,1} = \frac{\sqrt{1 - q\lambda^2}}{\lambda}\,\mathbf{e}_2,$$

$$\mathbf{E}_{3,1} = \frac{\sqrt{1 - r\lambda^2}}{\lambda}\,\mathbf{e}_3.$$

generate a factoring subalgebra for any parameters p, q, r.

The expressions $X_i(\lambda) = |\mathbf{E}_i|$ are functions on the elliptic curve

$$X_1^2 + p = X_2^2 + q = X_3^2 + r.$$

For any $i, k > 0$ there exists a unique element $\mathbf{E}_{ik} \in \mathcal{U}$ such that

$$\mathbf{E}_{ik} = \frac{\mathbf{e}_i}{\lambda^k} + O(k-1). \tag{29}$$

Let

$$L = \sum_{i,k} x_{ik}(t)\mathbf{E}_{ik}, \qquad A = \sum_{i,j} a_{ij}(t)\mathbf{E}_{ij}.$$

Consider the relation (so called Lax equation)

$$\frac{dL}{dt} - [A,\ L] = 0, \tag{30}$$

where $k \leqslant p$, $j \leqslant q$.

Lemma. This relation is equivalent to a finite system of ODEs for the coefficients x_{ik}, a_{ij}.

Proof. The l.h.s. is a series with finite number of terms of the form $P_i\lambda^{-i}, i > 0$. Suppose that all P_i equals zero. Then the l.h.s. is identically zero. Indeed, the $\frac{dL}{dt} - [A,\ L] \in \mathcal{U}$ and it is a Teylor series.

PDE case

In the PDE case the operator L in the Lax pair is not a matrix but an ordinary differential operator.

Any factoring subalgebra $\mathcal{U}$ in $\mathfrak{so}_3$ generates a Lax pair of the form

$$L = \frac{d}{dx} + U, \qquad U = \sum_{i=1}^{3} s_i \, \mathbf{E}_i, \qquad s_1^2 + s_2^2 + s_3^2 = 1, \quad (31)$$

$$A = \sum_i s_i \, [\mathbf{E}_j, \, \mathbf{E}_k] + \sum_i t_i \, \mathbf{E}_i, \quad (32)$$

leading to a nonlinear integrable PDE of the Landau–Lifschitz type. For this special case equation (30) can be written as

$$U_t - A_x + [U, \, A] = 0. \quad (33)$$

We can find the coefficients t_i and the corresponding nonlinear system of the form

$$\mathbf{s}_t = \vec{F}(\mathbf{s}, \, \mathbf{s}_x, \, \mathbf{s}_{xx}), \qquad \text{where} \quad \mathbf{s} = (s_1, s_2, s_3), \qquad \mathbf{s}^2 = 1,$$

using a direct calculation.

Namely, comparing the coefficients of λ^{-2} in the relation (33), we express t_i in terms of $\mathbf{s}, \mathbf{s}_x$. And then, equating the coefficients of λ^{-1}, we get a system of evolution equations for $\mathbf{s}$.

Example 1 (continuation). Equating to zero the coefficient of λ^{-2} in (33), we get $\mathbf{s}_x = \mathbf{s} \times \mathbf{t}$, where $\mathbf{t} = (t_1, t_2, t_3)$. Since $\mathbf{s}^2 = 1$, we find that $\mathbf{t} = \mathbf{s}_x \times \mathbf{s} + \mu\, \mathbf{s}$.

Comparing the coefficients of λ^{-1}, we arrive at the equation $\mathbf{s}_t = \mathbf{t}_x - \mathbf{s} \times \mathbf{V}\mathbf{s}$, where $\mathbf{V} = \operatorname{diag}(p, q, r)$. Substituting the expression for $\mathbf{t}$, we obtain

$$\mathbf{s}_t = \mathbf{s}_{xx} \times \mathbf{s} + \mu_x\, \mathbf{s} + \mu\, \mathbf{s}_x - \mathbf{s} \times \mathbf{V}\mathbf{s}.$$

Since the scalar product $(\mathbf{s},\, \mathbf{s}_t)$ has to be zero, we find that $\mu = \text{const}$. The resulting equation coincides (up to the involution $t \to -t$, a trivial additional term $\mu\, \mathbf{s}_x$ and a change of notation) with the Landau–Lifschitz equation

$$\mathbf{u}_t = \mathbf{u} \times \mathbf{u}_{xx} + \mathbf{R}(\mathbf{u}) \times \mathbf{u}, \qquad |\mathbf{u}| = 1.$$

Here $\times$ stands for the cross product.