$X/_k$ smooth proj conn.

char $k=0$

$\rightsquigarrow$ $H^\cdot(X)$ many flavors

$/k$ de Rham

$/\mathbb{Q}_\ell$ étale $k \hookrightarrow \bar{k}$

$/\mathbb{Q}$ Betti $k \hookrightarrow \mathbb{C}$

$\supset H^\cdot_{alg}(X)/_{\mathbb{Q}} \simeq \mathbb{Q}^r \simeq \mathbb{Q} \otimes K_0(D^b\mathrm{Coh}X)/_{\sim\mathrm{hom}}$

$r = r_0 + r_1 + \dots + r_{\dim X}$

$r_0 = 1,\ r_1 = \mathrm{rk}\, NS(X) \dots$

Standard Conj.

$\Rightarrow r_i = r_{\dim X - i}$

Do not assume!

Gromov-Witten inv. genus $=0$:

multiplication on $H^\cdot_B(X)$

depending on a point in "domain" in $H^\cdot_B(X)/H^2_B(X,\mathbb{Z})$

analytic

$K = \mathbb{Q}((T))$

dummy variable

$\{\Delta_\alpha\}$ graded basis of $H^\cdot(X)$

t_0 corr. to $1 \in H^0$

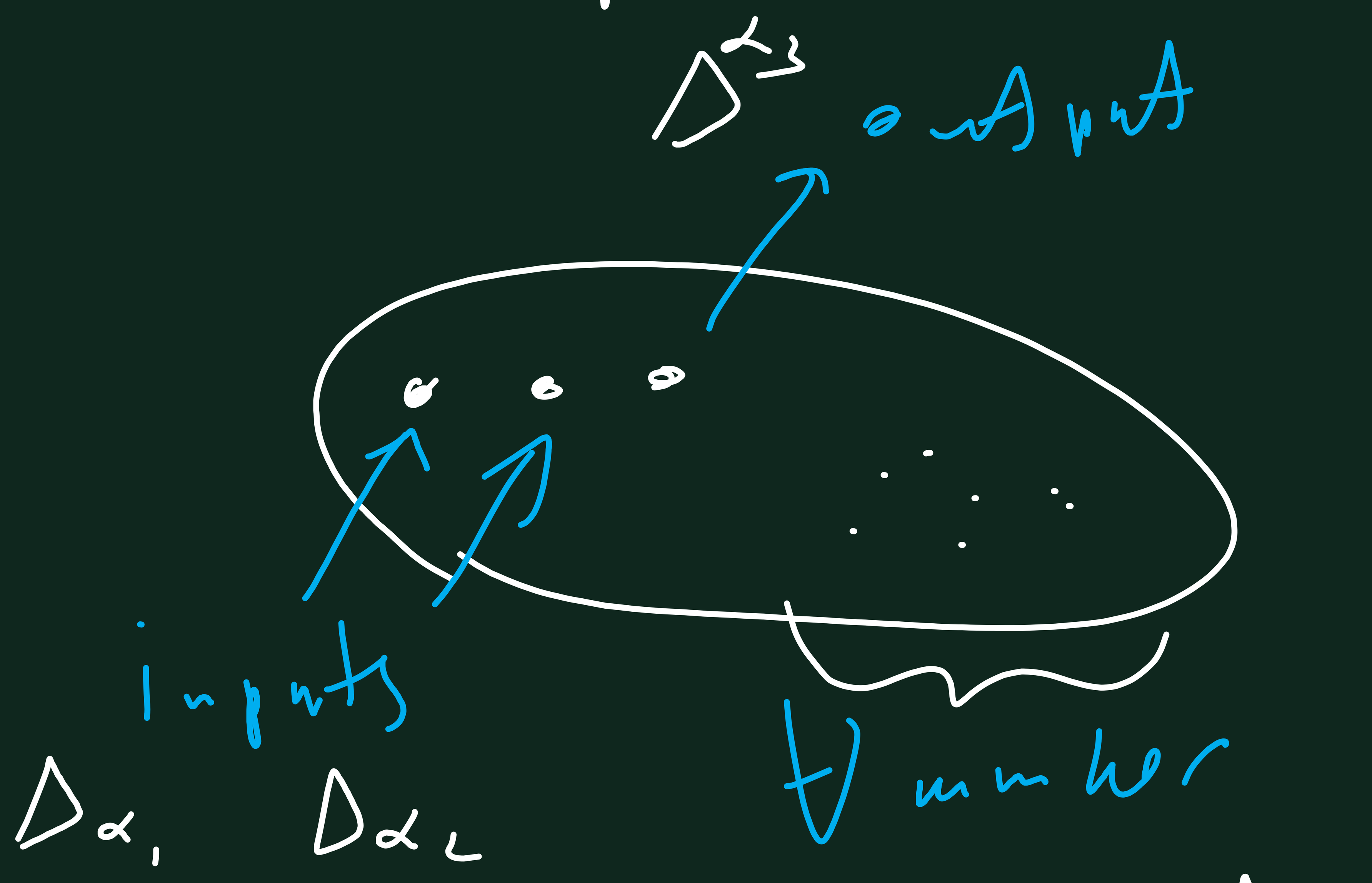
(q_α) deg $\Delta_\alpha = 2$, (t_α) deg $\Delta_\alpha \neq 0, 2$

$q_\alpha = e^{t_\alpha}$

multiplication / $\mathbb{Q}[[(q_\alpha)_{\deg \Delta_\alpha = 2}, (t_\alpha)_{\deg \Delta_\alpha \neq 0,2}]]$

+ trivial dep on t_0 $\quad \deg \Delta_0 = 0$

$\quad 1$



$$\sum_{\deg \Delta_\alpha \neq 0,2} t_\alpha \Delta_\alpha$$

$$\cdot \prod_{\alpha : \deg \Delta_\alpha = 2} q_\alpha^{\int_C \Delta_\alpha}$$

(Δ_α) ample classes basis of NS

$q_\alpha, t_\alpha \in K = \overline{\mathbb{Q}((T))}$

$0 < |q_\alpha| < 1, \quad |t_\alpha| < 1$

tube domain

$(K^\times)^{r_1}$

$\downarrow$

$\mathbb{R}^{r_1} \supset$ — ample cone

$\Rightarrow \mathcal{F}_X$ analytic$/K$

F-bundle

$\mathcal{H}$ bundle

∇ merom conn.

fiber $= H^{\cdot}(X)$

$$B = F_X \times \operatorname{Spec} K[[u]]$$

$$\nabla_{(\partial_{t_\alpha} \text{ or } q_\alpha \partial_{q_\alpha})} = \partial_{t_\alpha \text{ or }} \partial_{q_\alpha \partial_{q_\alpha}} + \frac{1}{u}(\Delta_\alpha * \cdot)$$

$$\nabla_{\partial_u} = \partial_u + u^{-2}\, \mathbb{K} + u^{-1} \mathrm{Gr}$$

$$\mathbb{K} = \Big(c_1(T_X) + \sum_{\deg \Delta_\alpha \neq 2} \frac{\deg \Delta_\alpha - 2}{2} \cdot \Delta_\alpha \Big) \times \cdot$$

Gr : acts on $H^{i}(X)$ as $\dfrac{i - \dim X}{2}$

0) $\nabla^2 = 0$

1) ∇ has 1st order pol along B

$$T_b B \longrightarrow \operatorname{End}(H_{b,0})$$

comm. operators.

Axiom: $\exists\, v \in H_{b,0}$ action $b \to$ iso $T_b B \simeq H_{b,0}$

$\Rightarrow$ $T_b B$ assoc unital comm. alg.

2) ∇ has 2nd order pole along u

$\leadsto$ $\mathbb{K} = E u \cdot$ $\quad E u$: Euler field

For $\forall$ F-bundle $/ B$ (e.g. coming from GW th. $B = F_X$)

$b \in B \rightsquigarrow \mathbb{K}_b$ operator $\longrightarrow \operatorname{Spec}(\mathbb{K}_b) \in \dfrac{\text{ass. f. d. 1}}{\mathbb{Q}((T))}$

$\# \operatorname{Spec}(\mathbb{K}_b) \in \mathbb{Z}_{>0}$

constr. function — maximal value: analog of Milnor #.

"Lyashko-Looijenga" stratification

Example: f: germs of $\mathbb{C}$ an. function isolated singul.

B = germ of universal unfolding

$f = x^{n+1} + a_1 x^{n-1} + \dots + a_n$

$\mathbb{K}$: $f \cdot$ in $\mathcal{O}(\operatorname{Crit} f)$

dense open domain — $\# \operatorname{Spec}(\mathbb{K}_b)$ is maximal

semisimple

"atoms" := eigenvalues at the generic locus

$F_X \supset$ subset $F_{X, \mathrm{alg}}$ "alg. atoms"

$\#$(eigenvalues of $\mathbb{K}$. is maximal).

$t_0 1 + \ldots$

Example (1) if X is CY, gen. type ...

$\int c_1(K_X) \geq 0$

$\forall$ red. curve

$$\boxed{K_X \geq 0} \implies \mathrm{Spec}\,\mathbb{K} = \{t_0\}$$

if $t_0 = 0$: $\mathbb{K}_X$ strictly increases deg in H.

$\implies$ only one atom.

(2) $X = \mathbb{P}^N$. $q = 1$ $t = 0$

eigenvalues $\sqrt[N+1]{-1}$

$N+1$

all atoms are "the same" as for $X = pt$

Conj (2019)

near this root

open domain in $F_{Bl_Z X}$

$\cong$ canonical nbhd $\prod$ open domains in $F_X, \underbrace{F_{Z}, \dots, F_Z}_{d-1 \text{ times}}$

$Bl_{Z\times\{0\}}(X\times\mathbb{P}^1)$

$\mathbb{G}_m$-action

H. Iritani: Proved it!

Blow-up formula

$X \supset Z$ closed smooth

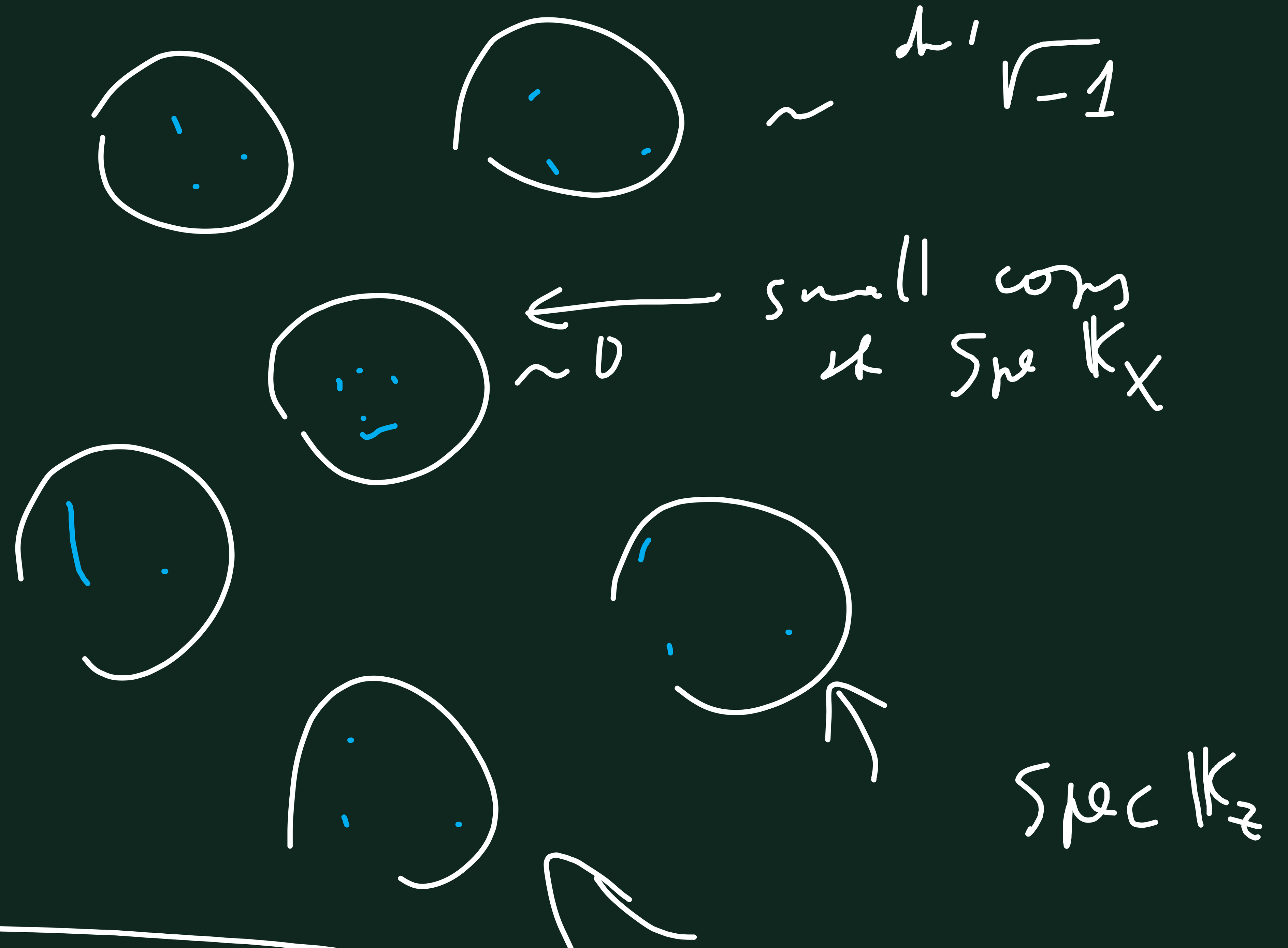
pure codim $= d \geq 2$

$\dashrightarrow Bl_Z X$

point $F_{Bl_Z X}$

close to $(t)=0$

q: ample class close to pull back of ample on X



Corollary: very general cubic 4-fold $X \subset \mathbb{P}^5 / \mathbb{C}$ is not rational.

$H^4(X)$ contains only one $/\mathbb{Q}^{+}$ Hodge class

$\Rightarrow \operatorname{rk} H_{alg}(X) = 5$

Refor $*$ mult. along $H^4(X)$

A. Givental

$\circ \leftarrow 1\text{-d}$

$\circ\ 1\text{-d}$

$\circ\ 1\text{-}n$

G_{MT} Mumford-Tate group pro-reductive $/\mathbb{Q}$

$\otimes$ polarizable pure Hodge str.

$G^{\circ}_{MT} = \operatorname{Ker}(G_{MT} \to \mathbb{G}_m)$

action $H^2(\mathbb{P}^1)$

$\forall\ G^{\circ}_{MT}$ rep. $(p-q)$ decomposition $H^{p,q}$

$\operatorname{rk} H^{3,1}(X) = 1$

repr. of $G^{\circ}_{MT, \mathbb{Q} \to \bar{\mathbb{Q}}}$

$\operatorname{rk}(p-q) = 2$ part $= 1$

$\operatorname{rk}(\text{place})^{G^{\circ}_{MT}} = 2$

$\Rightarrow$ ~~one~~ of atoms $\operatorname{rk}(p-q) \geq 1$, $\operatorname{rk}(\quad)^{G^{\circ}} \leq 2$

Blow-ups at pt, curves, surfaces

$p-q=c$ is absent

Classification

$\forall$ surface S

rk $\Gamma(S, K_S) \geq 1$

$\exists$ bi. model S'

$K_{S'} \geq 0$

Abstract def of atoms for a given field k:

X

$F_{X,\mathrm{alg}} \to \mathrm{Disc}_{/\overline{\mathbb{Q}((T))}}$

$\uparrow$ étale map

$\{\text{eigenvalue of } \mathbb{K}\}$

$\mathrm{loc.\ atoms}(X) = \pi_0(\text{total space})/\mathrm{Aut}\,X$

$\# < \infty$

Can have $p-q=c$ piece

and rk $H^i(S')^{G_{MT}} \geq 3$

$\mathrm{Atoms}_k = \coprod_X \mathrm{loc.\ atoms}(X)/\sim$

Identify

Atoms $\mathrm{Bl}_Z(X)$

with $\mathrm{loc.atoms}(X)$, (Z)

$\begin{matrix} \mathcal{E} \\ \downarrow \text{vect.} \\ X \end{matrix}_k$ / $\mathrm{Atom}(\mathbb{P}(\mathcal{E})) \sim \mathrm{Atom}(X)$

$\vdots$

Using product of varieties
F-bundles

: mult. p! on $\mathbb{Z}^{\mathrm{Atoms}_k}$

com. ass. unital ring

structure constant $\in \mathbb{Z}_{\geq 0}$.

$$K_0(\mathrm{Var}_k) \longrightarrow \mathbb{Z}^{\mathrm{Atoms}_k}$$

Bittner rel   ring homom.

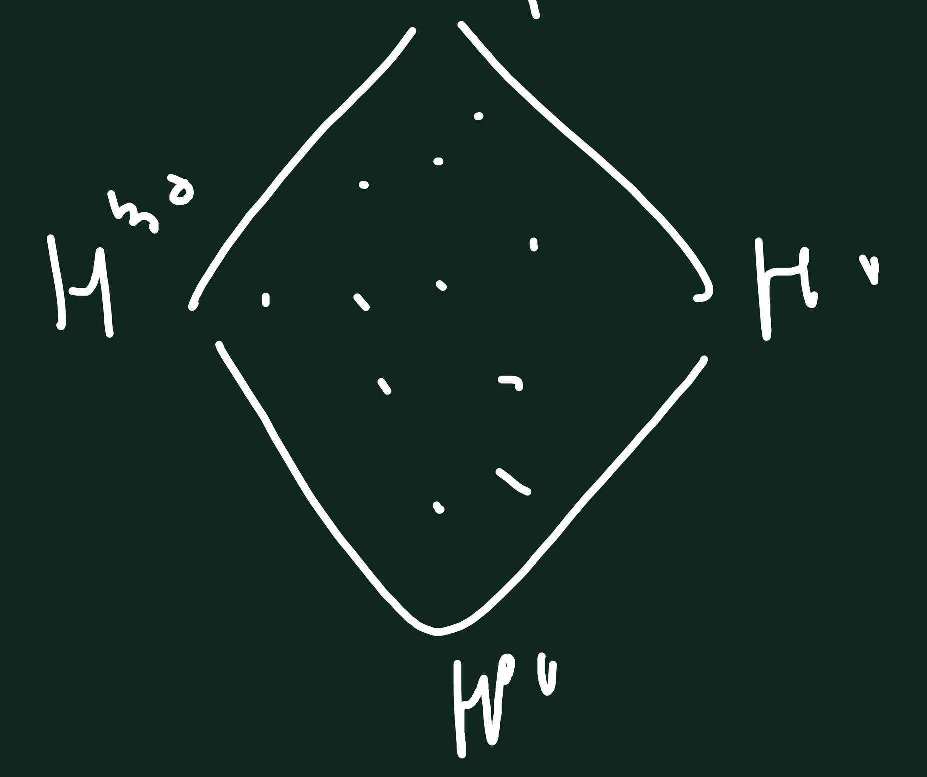
$$\mathrm{Atom}_k = \bigcup (\mathrm{Atom}_k)_{\leq n}$$

$n = \dim X$

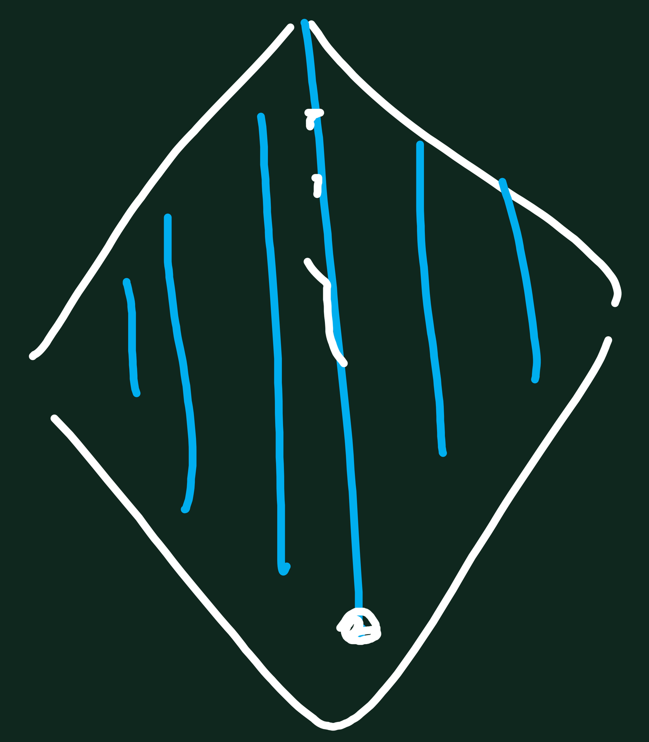
if X ($\dim X = n$) has atom which does not come from $\leq (n-2)$ varieties (one repres. in each bir. class)

$\Rightarrow X$ is not rational.

? $\forall$ Atom, Hodge # 4



Atom



$p - q \in \mathbb{Z}$

$p + q \in \mathbb{Q}$

? Convergence / $\mathbb{C}$

for alg. varieties

? more manageable

x $|u| \ll 1$

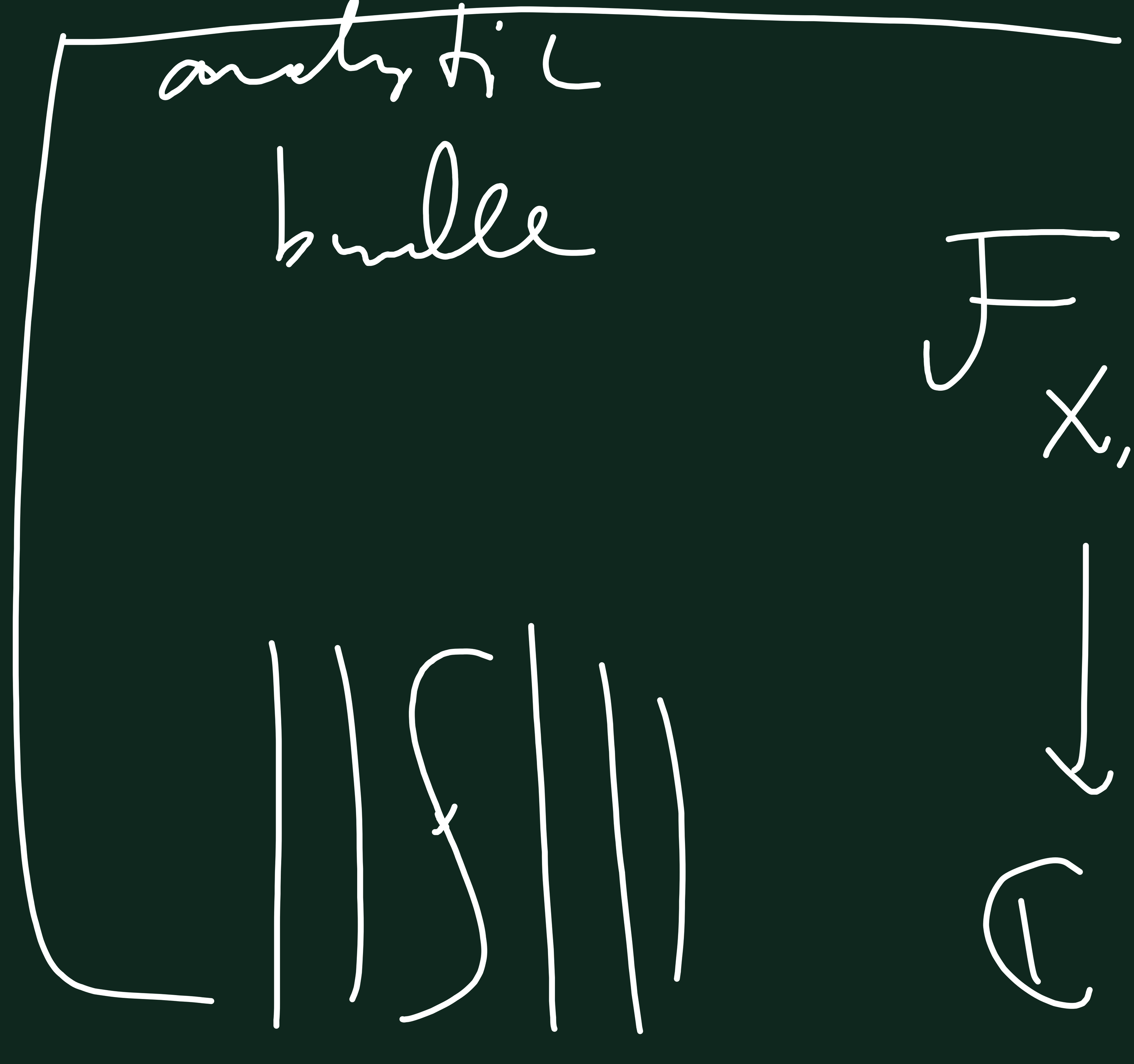
isol. sing

X^{mod}

$+\frac{1}{2}$

$-\frac{1}{2}$

defo.



ser: tube dom. — ample cone

analytic bundle

$$\mathcal{F}^{\text{ext}}_{X,\text{ug}} \xrightarrow{u} \mathbb{C} \quad \text{submersion.}$$

fiber at $u=0$ = $\mathcal{F}_{X,\text{ch}}$

fibre at $u \neq 0$ domain in a vector space $H^{\cdot}(X;\mathbb{C})$ at pt $q>0$ too

monodromy: $e^{c_1(\text{line b})}$ $u>0$, $e^{c_1(T_X)}$.

Conj: some open nbhd $\hookrightarrow \mathrm{Stab}(D^b \mathrm{Coh}\, X)$: