# Novikov-Veselov symmetries of $O(N)$ sigma-model 

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## Two-dimensional sigma-models

Harmonic maps of two-dimensional Riemann surface $\Sigma$ to a Riemann manifold $M$ are of interest both in physics and mathematics. They are critical points of the Dirichlet functional, the sigma model action

$$
\begin{equation*}
S(X)=\int_{\Sigma} \sqrt{h} h^{a b} g_{i j}(X) \partial_{a} X^{i} \partial_{b} X^{j} d x d y=\int_{\Sigma} g_{i j}(X) \partial X^{i} \bar{\partial} X^{j} \tag{1}
\end{equation*}
$$

Here the map $\Phi: \Sigma \rightarrow M$ is represented, locally, by the pullbacks $X^{i}(x, y)=\Phi^{*} x^{i}, i=1, \ldots, \operatorname{dim} M$, of the coordinate functions ( $x^{i}$ ) on $M$ to $\Sigma$, with $x, y$ local real coordinates on $\Sigma$ and $(z, \bar{z})$ denote the complex coordinates on $\Sigma$, in the complex structure determined by the conformal class of the metric $h_{a b}$ via $h_{a b} d y^{a} d y^{b} \propto d z d \bar{z}$. Finally, $g_{i j}(X) d X^{i} d X^{j}$ is a Riemann metric on the target manifold $M$.

## $O(N)$ sigma model

The target manifold is the unit sphere, $M=S^{N-1}$, with induced metric.

- Equations of motion

$$
\begin{equation*}
\left(-\partial_{z} \partial_{\bar{z}}+u(z, \bar{z})\right) q^{i}(z, \bar{z})=0 \tag{2}
\end{equation*}
$$

The potential $u$ is the Lagrange multiplier enforcing the constraint that the vector $q \in \mathbb{R}^{N}$ with the coordinates $q^{i}$ lies on the unit square

$$
\begin{equation*}
(q, q)=1 \tag{3}
\end{equation*}
$$

From (2) and (3) it is easy to get:

$$
\begin{equation*}
u=-\left(\partial_{z} a \cdot i_{\bar{z}} q\right) \tag{4}
\end{equation*}
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Most of our results are evenly applicable to the case of arbitrary quadric


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(q, q):=\sum_{i j} \eta_{i j} q^{i} q^{j}
$$

## Spectral curves from zero-curvature representation

Our main interest is in the double-periodic two-dimensional sigma model, i.e. we take $\Sigma$ to be a two-dimensional torus $T^{2}=S^{1} \times S^{1}$. Its conformal structure is parameterized by the complex number $\tau$, with $\Im \tau>0$.

The harmonic maps of the two-torus $T^{2}$ to $S^{3}$ were constructed by Hitchin via the zero-curvature representation for the principal chiral $S U(2)$ model. The latter is the compatibility condition for the system of two linear equations


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$$
\begin{equation*}
\left(\partial_{z}-\frac{U(z, \bar{z})}{\lambda+1}\right) \Psi(z, \bar{z}, \lambda)=0, \quad\left(\partial_{\bar{z}}+\frac{V(z, \bar{z})}{\lambda-1}\right) \Psi(z, \bar{z}, \lambda)=0 \tag{5}
\end{equation*}
$$

with $U=X^{-1} \partial_{z} X$ and $V=X^{-1} \partial_{\bar{z}} X$.

The key result of Hitchin was a proof that the branch points of the two-sheet cover of the complex $\lambda$ plane defined by the characteristic equations

$$
\begin{equation*}
R_{\alpha}\left(\mu_{a}, \lambda\right):=\operatorname{det}\left(\mu_{\alpha} \cdot \mathbb{I}-B_{\alpha}(\lambda)\right)=0 \tag{6}
\end{equation*}
$$

for the monodromy matrices

$$
\begin{equation*}
B_{\alpha}(\lambda):=\Psi\left(z+\omega_{\alpha}, \bar{z}+\bar{\omega}_{\alpha}, \lambda\right) \Psi^{-1}(z, \bar{z}, \lambda), \alpha=1,2 \tag{7}
\end{equation*}
$$

coincide and there number is finite.
The hyperelliptic curve defined by these branch points is a normalization of the analytic spectral curves (6).

The serious drawback of the inverse reconstruction of the harmonic map from the hyperelliptic curve and a point of its Jaconian is the periodicity constraint. In general the reconstruction gives quasi-periodic maps of the universal cover of $\Sigma$. The equations that single out periodic maps are given in terms of periods of certain abelian differentials of the second kind on the hyperelliptic curves. The equations are transcendental and hard to control. Hitchin proved that there are solutions to these equations for hyperelliptic curves of genera 1,2,3, only.

## Harmonic maps to $S^{2}$

For further comparison with the results of that work, one more observation made by Hitchin should be emphasized:
there is only one class of harmonic maps which can not be reconstructed by algebraic-geometrical data, and those are the ones for which the Floquet multipliers $\mu_{\alpha}$ are constants.

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These maps are known explicitly (Polyakov, Belavin)

$$
\begin{gathered}
x+i x_{2}=\frac{2 w}{1+|w|^{2}}, \quad x_{3}=\frac{1-|w|^{2}}{1+|w|^{2}} \\
w(z)=\frac{\prod_{i=1}^{\ell} \sigma\left(z-a_{i}\right)}{\prod_{i=1}^{\ell} \sigma\left(z-b_{i}\right)}, \quad \sum_{i} a_{i}=\sum_{i} b_{i}
\end{gathered}
$$

The NV hierarchy was introduced as the compatibility condition of the system of linear equations

$$
\begin{gather*}
H \psi:=\left(-\partial_{z} \partial_{\bar{z}}+u\right) \psi=0  \tag{8}\\
\left(\partial_{t_{n}}-L_{n}\right) \psi=0 \tag{9}
\end{gather*}
$$

where $L_{n}$ is a differential operator of the form

$$
\begin{equation*}
L_{n}=\partial_{z}^{2 n+1}+\sum_{i=1}^{2 n-1} w_{i, n}(z, \bar{z}) \partial_{z}^{i} \tag{10}
\end{equation*}
$$

The compatibility condition of linear equations $(8,9)$ is the, so-called, Manakov's triple equation

$$
\begin{equation*}
\partial_{t_{n}} H=\left[L_{n}, H\right]+B_{n} H \tag{11}
\end{equation*}
$$

where $B_{n}$ is a differential operator in the variable $z$.

## The phase space

The NV hierarchy can be defined as a system of commuting flows on a space $\mathcal{P}$ of self-dual wave (formal Baker-Akhiezer) solutions of the Schrödinger equation (8), i.e. the formal solutions of the form

$$
\begin{equation*}
\psi(z, \bar{z}, k)=e^{k z}\left(1+\sum_{s=1}^{\infty} \xi_{s}(z, \bar{z}) k^{-s}\right) \tag{12}
\end{equation*}
$$

such that equations

$$
\begin{equation*}
\operatorname{Res}_{\infty}\left(\psi(z, \bar{z},-k) \partial_{z}^{s} \psi(z, \bar{z}, k)\right) \frac{d k}{k}=-\delta_{s, 0}, \quad s=0,1, \ldots \tag{13}
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holds.

$$
\begin{gathered}
\mathcal{P}=\left\{u(z, \bar{z}), \chi_{1}(z), \chi_{3}(z), \ldots\right\} \\
\chi_{s}(z):=\xi_{s}(z, 0)
\end{gathered}
$$

## The flows

Given the formal BA series $\psi(z, \bar{z}, k)$ define:
(i) the wave operator

(ii) the pseudo-differential operator


## (iii) the differential operator


such that


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$$

(iii) the differential operator

$$
\mathcal{L}_{+}^{n}=\sum_{i=1}^{n} F_{n}^{(-i)} \partial_{z}^{i}
$$

such that

$$
\begin{gathered}
\mathcal{L}_{-}^{n}=\mathcal{L}^{n}-\mathcal{L}_{+}^{n}=F_{n}^{(0)}+F_{n}^{(1)} \partial_{z}^{-1}+O\left(\partial_{z}^{-2}\right) \\
F_{n}^{(1)}=\operatorname{Res}_{\partial_{z}} \mathcal{L}^{n}
\end{gathered}
$$

## The flows II

Now we are ready to define NV hierarchy explicitly.

## Theorem

The equations

$$
\begin{gather*}
\partial_{t_{n}} u=\partial_{\bar{z}} F_{2 n+1}^{(1)},  \tag{14}\\
\partial_{t_{n}} \psi=\mathcal{L}_{-}^{2 n+1} \psi \tag{15}
\end{gather*}
$$

define a family of commuting flows on the space $\mathcal{P}$ of self-dual formal BA solutions of the Schrödinger operators.

## Theorem

Let $q(z, \bar{z})$ be a solution of the $O(N)$ sigma model. Then there exists a unique up to multiplication by $(z, \bar{z})$-independent factor, i.e.

$$
\begin{equation*}
\psi \longmapsto \psi \rho(k), \quad \rho(k)=\exp \left(\sum_{s=1}^{\infty} \rho_{s} k^{-2 s+1}\right) \tag{16}
\end{equation*}
$$

self-dual formal BA solution $\psi$ of the Schrödinger equation with the potential u given by (4) such that the constraint (3) is invariant under the commuting flows

$$
\begin{equation*}
\partial_{t_{n}} q=\mathcal{L}_{+}^{2 n+1} q \tag{17}
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Essentially, the proof of the theorem is the proof of the equality


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$$
\left(\mathcal{L}_{+}^{2 n+1} q, q\right)=0
$$

## $O(N)$ sigma model on a two-torus

## Lemma

Let $u(z, \bar{z})$ be a double periodic function. Then there is a unique up to the transformation (16) self-dual BA formal solution of the Schrödinger equation (2) of the form

$$
\begin{equation*}
\psi(z, \bar{z}, k)=e^{k z+\ell(k) \bar{z}}\left(1+\sum_{s=1}^{\infty} \zeta_{s}(z, \bar{z}) k^{-s}\right) \tag{18}
\end{equation*}
$$

with double periodic coefficients $\zeta_{s}$, i.e.

$$
\begin{equation*}
\zeta_{s}\left(z+\omega_{\alpha}, \bar{z}+\bar{\omega}_{\alpha}\right)=\zeta_{s}(z, \bar{z}), \alpha=x, y \tag{19}
\end{equation*}
$$

and where $\ell(k)$ is a formal series,

$$
\begin{equation*}
\ell(k)=\sum_{s=1}^{\infty} \ell_{s} k^{-s} \tag{20}
\end{equation*}
$$

## Theorem

The equations

$$
\begin{equation*}
\partial_{t_{n}} u=\partial_{\bar{z}} F_{2 n+1}^{(1)} \tag{21}
\end{equation*}
$$

with $F_{2 n+1}^{(1)}=\operatorname{Res}_{\partial} \mathcal{L}_{+}^{2 n+1}$, where $\mathcal{L}$ is defined by the formal $B A$ solution defined above are commuting flows on the space of double periodic functions.
Moreover if $u=-\left(\partial_{z} q, \partial_{\bar{z}} q\right)$ for some solution of $O(N)$-sigma model then the constraint $(q, q)=1$ are preserved by the flows

$$
\begin{equation*}
\partial_{t_{n}} q=\mathcal{L}_{+}^{2 n+1} q \tag{22}
\end{equation*}
$$

## The algebraic spectral curve

Let $q(z, \bar{z}, t), t=\left(t_{1}, t_{2}, \ldots\right)$ be a an orbit of $q(z, \bar{z}, 0)$ under the flows of the NV hierarchy. Taking the $t_{n}$ derivative of (2) and using the equation

$$
\begin{equation*}
\partial_{t_{n}} u=-\left(\partial_{z}\left(\mathcal{L}_{+}^{2 n+1} q\right), \partial_{\bar{z}} q\right)-\left(\partial_{z} q, \partial_{\bar{z}}\left(\mathcal{L}_{+}^{2 n+1} q\right)\right) \tag{23}
\end{equation*}
$$

we get the equation

$$
\mathfrak{D}\left(\mathcal{L}_{+}^{2 n+1} q\right)=0
$$

where

$$
\mathfrak{D}=\partial_{z} \partial_{\bar{z}}+\left(q \otimes\left(\partial_{\bar{z}} q\right)^{t}\right) \partial_{z}+\left(q \otimes\left(\partial_{z} q\right)^{t}\right) \partial_{\bar{z}}+\left(\partial_{z} q, \partial_{\bar{z}} q\right)
$$

is an elliptic operator on $\Sigma$.

An elliptic equation on $\Sigma$ has only finite number of solutions. Hence for all but a finite number of integers $n$ there are constants $c_{n, m}$ such that for the operator

$$
\begin{equation*}
\tilde{L}_{n}:=\mathcal{L}_{+}^{2 n+1}+\sum_{i=0}^{n-1} c_{n, m} \mathcal{L}_{+}^{2 m+1} \tag{24}
\end{equation*}
$$

the equation

$$
\begin{equation*}
\tilde{L}_{n} q=0 \tag{25}
\end{equation*}
$$

holds.

## Definition

A Schrödinger operator is called algebraic-geometric (finite-gap) if it is stationary for all but a finite number of the NV hierarchy flows.

The operators $\tilde{L}_{n}$ defined above commute with each other

$$
\begin{equation*}
\left[\tilde{L}_{n}, \tilde{L}_{m}\right]=0 \tag{26}
\end{equation*}
$$

## Lemma (Burchnall-Chundy)

Let $L_{n}$ and $L_{m}$ be commuting ordinary linear differential operators of orders $n$ and $m$, respectively. Then there exists a polynomial $R$ in two variables such that the equation

$$
R\left(L_{n}, L_{m}\right)=0
$$

## holds.

The affine curve defined by equation (27) is compactified by one smooth point $P_{+}$
The corresponding algebraic curve $\Gamma$ is called the spectral curve.

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For a generic pair of commuting operators of co-prime orders the spectral curve is smooth and the common eigenfunction $\psi$ of the commuting operators is the Baker-Akhiezer function:
> $1^{0}$. as a function of $p \in \Gamma$ it is meromorphic on $\Gamma \backslash P_{+}$with $z$-independent divisor $D$ of poles of degree equals the genus of
> $2^{0}$. in the neighborhood of $P_{+}$it has the form (12).
> A priory in the problem under consideration there are two spectral curves. One, which we have just discussed, and the second one with marked smooth point $P_{-}$which corresponds to commuting operators in the variable $\bar{z}$. In fact, these spectral curves coincide.

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## Theorem

The spectral curve $\Gamma$ of an algebraically integrable operator $H$ is a curve with involution. In the generic case when $\Gamma$ is smooth: the points $P_{ \pm}$are the only fixed points of $\sigma$; the common eigenfunction $\psi$ of the operators $L_{n}, \bar{L}_{m}$ satisfying the equation $H \psi=0$ in the neighborhoods of $P_{ \pm}$it has the form

$$
\begin{aligned}
& \psi(z, \bar{z}, p)=e^{k_{+} z}\left(1+\sum_{n=1}^{\infty} \xi_{n}^{+}(z, \bar{z}) k_{+}^{-n}\right), p \rightarrow P_{+} \\
& \psi(z, \bar{z}, p)=e^{k_{-} \bar{z}}\left(1+\sum_{n=1}^{\infty} \xi_{n}^{-}(z, \bar{z}) k_{-}^{-n}\right), p \rightarrow P_{-}
\end{aligned}
$$

and outside marked points $P_{ \pm}$it is meromorphic with $(z, \bar{z})$-independent divisor of poles $D$ satisfying the constraint

$$
\begin{equation*}
D+D^{\sigma}=\mathcal{K}+P_{+}+P_{-} \tag{28}
\end{equation*}
$$

where $\mathcal{K}$ is the canonical class, i.e. the equivalence class of the

## Periodicity constraint

The potential $u(z, \bar{z})$ is periodic if and only if there are functions $w_{\alpha}^{ \pm}$on $\Gamma_{ \pm}$such that equations

$$
\begin{equation*}
\psi\left(z+\omega_{\alpha}, \bar{z}+\bar{\omega}_{\alpha}, p\right)=w_{\alpha}(p) \psi_{ \pm}(z, \bar{z}, p) \tag{29}
\end{equation*}
$$

hold.
The differential $d p_{\alpha}=d \ln w_{\alpha}$ is a meromorphic differential on $\Gamma$ with the poles at $P_{ \pm}$of the form

$d p_{x}=d k_{-}\left(1+O\left(k_{-}^{-2}\right)\right), \quad d p_{y}^{-}=\bar{\tau} d k_{-}\left(1+O\left(k_{-}^{-2}\right)\right)$
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\begin{array}{ll}
d p_{x}=d k_{+}\left(1+O\left(k_{+}^{-2}\right)\right), & d p_{y}^{+}=\tau d k_{+}\left(1+O\left(k_{+}^{-2}\right)\right) \\
d p_{x}=d k_{-}\left(1+O\left(k_{-}^{-2}\right)\right), & d p_{y}^{-}=\bar{\tau} d k_{-}\left(1+O\left(k_{-}^{-2}\right)\right),
\end{array}
$$

The definition of $d p_{\alpha}$ implies

$$
\begin{equation*}
\oint_{c} d p_{\alpha} \in 2 \pi i \mathbb{Z} \quad \forall c^{ \pm} \in H_{1}(\Gamma, \mathbb{Z}) \tag{30}
\end{equation*}
$$

## $w_{\infty}$-harmonic maps

Note that if the equations

$$
\begin{equation*}
\left(\partial_{z}^{j} q, \partial_{z}^{j} q\right)=0, \quad 0<j<m \tag{31}
\end{equation*}
$$

hold then using the Shrödinger equation it is easy to see that $\partial_{\bar{z}}\left(\partial_{z}^{m} q, \partial_{z}^{m} q\right)=0$. The model is invariant under a conformal change of variables $z \mapsto f(z), \bar{z} \mapsto \bar{f}(\bar{z})$. Hence if $\left(\partial_{z}^{m} q, \partial_{z}^{m} q\right) \neq 0$, then without loss of generality we may assume that

$$
\begin{equation*}
\left(\partial_{z}^{m} q, \partial_{z}^{m} q\right)=(-1)^{m+1} \tag{32}
\end{equation*}
$$

We will call such solutions $w_{m}$-harmonic. Notice, that for $m>1$ they are conformal.

> Recall, conformal harmonic maps are maps for which the pull-back $X^{*}(g)$ of the target space metric $g$ is conformally equivalent to the worldsheet metric $h$ on $\Sigma$. They are of special interest in geometry since their images are immersed minimal

## $w_{\infty}$-harmonic maps

Note that if the equations

$$
\begin{equation*}
\left(\partial_{z}^{j} q, \partial_{z}^{j} q\right)=0, \quad 0<j<m \tag{31}
\end{equation*}
$$

hold then using the Shrödinger equation it is easy to see that $\partial_{\bar{z}}\left(\partial_{z}^{m} q, \partial_{z}^{m} q\right)=0$. The model is invariant under a conformal change of variables $z \mapsto f(z), \bar{z} \mapsto \bar{f}(\bar{z})$. Hence if $\left(\partial_{z}^{m} q, \partial_{z}^{m} q\right) \neq 0$, then without loss of generality we may assume that

$$
\begin{equation*}
\left(\partial_{z}^{m} q, \partial_{z}^{m} q\right)=(-1)^{m+1} \tag{32}
\end{equation*}
$$

We will call such solutions $w_{m}$-harmonic. Notice, that for $m>1$ they are conformal.

Recall, conformal harmonic maps are maps for which the pull-back $X^{*}(g)$ of the target space metric $g$ is conformally equivalent to the worldsheet metric $h$ on $\Sigma$. They are of special interest in geometry since their images are immersed minimal surfaces.

## Reducible spectral curves

Let $\Gamma_{ \pm}$be a smooth genus $g_{ \pm}$algebraic curve with the holomorphic involution

$$
\begin{equation*}
\sigma: \Gamma_{ \pm} \longmapsto \Gamma_{ \pm} \tag{33}
\end{equation*}
$$

with $2(1+n)$ fixed points

$$
\begin{equation*}
\sigma\left(P_{ \pm}\right)=P_{ \pm}, \sigma\left(p_{ \pm}^{i}\right)=p_{ \pm}^{i} \tag{34}
\end{equation*}
$$

Let us fix the $\sigma$-odd local parameters ${k_{ \pm}^{-1}}^{\text {in }}$ the neighborhoods of the marked points $P_{ \pm}$,

$$
\begin{equation*}
k_{ \pm}(\sigma(p))=-k_{ \pm}(p) \tag{35}
\end{equation*}
$$

The projection

$$
\begin{equation*}
\pi: \Gamma_{ \pm} \longmapsto \Gamma_{ \pm}^{0}=\Gamma_{ \pm} / \sigma \tag{36}
\end{equation*}
$$

represents $\Gamma_{ \pm}$as a two-sheet covering of the quotient-curve $\Gamma_{ \pm}^{0}$ with $2(n+1)$ branch points $P_{ \pm}, p_{ \pm}^{i}$, the involution $\sigma$ permuting the sheets.

## The divisors

Let $d \Omega_{ \pm}(p)$ be a third kind meromorphic differential on $\Gamma_{ \pm}^{0}$ with the divisor of poles at the branching locus ( $\left.\Gamma_{ \pm}\right)^{\sigma}$ with residues $\mp 1$ at the marked points $P_{ \pm}$. The differential $d \Omega_{ \pm}$has $g_{ \pm}+n$ zeros that we denote by $\gamma_{s}^{0, \pm}$,

$$
\begin{equation*}
d \Omega_{ \pm}\left(\gamma_{s}^{0, \pm}\right)=0 . \tag{37}
\end{equation*}
$$

For each zero $\gamma_{s}^{0, \pm}$ we choose one of its preimages on $\Gamma_{ \pm}$, i.e. a point $\gamma_{s}^{ \pm}$on $\Gamma_{ \pm}$such that

$$
\begin{equation*}
\pi\left(\gamma_{s}^{ \pm}\right)=\gamma_{s}^{0, \pm}, s=1, \ldots, g_{ \pm}+n . \tag{38}
\end{equation*}
$$

(there are $2^{g_{ \pm}+n}$ such choices). Below $D_{ \pm}=\gamma_{1}^{ \pm}+\ldots+\gamma_{g_{ \pm}+n}^{ \pm}$ will be called the admissible divisor.

For a generic set of data ( $\left.\Gamma_{ \pm}, \sigma, P_{ \pm}, k_{ \pm}, p_{ \pm}^{i}, \gamma_{s}^{ \pm}\right)$and a matrix $G \in O(2 n+1, C)$, there is a unique pair of functions $\psi_{ \pm}(z, \bar{z}, p)$ on $\Gamma_{ \pm}$such that:
$1^{0}$. Outside $P_{ \pm}$the function $\psi_{ \pm}$is meromorphic with the pole divisor $D_{ \pm}$;
$2^{0}$. In the neighborhoods of $P_{ \pm}$the function $\psi_{ \pm}$has the form

$$
\begin{aligned}
& \psi(z, \bar{z}, p)=e^{k_{+} z}\left(1+\sum_{n=1}^{\infty} \xi_{n}^{+}(z, \bar{z}) k_{+}^{-n}\right), p \rightarrow P_{+} \\
& \psi(z, \bar{z}, p)=e^{k_{-} \bar{z}}\left(1+\sum_{n=1}^{\infty} \xi_{n}^{-}(z, \bar{z}) k_{-}^{-n}\right), p \rightarrow P_{-}
\end{aligned}
$$

$3^{0}$. The gluing equations

$$
\begin{equation*}
y_{+}(z, \bar{z})=G y_{-}(z, \bar{z}) \tag{39}
\end{equation*}
$$

where $y_{ \pm}$are vectors with the coordinates

$$
\begin{gather*}
y_{ \pm}^{i}=r_{ \pm}^{i} \psi_{ \pm}\left(z, \bar{z}, p_{ \pm}^{i}\right), \quad i=1, \ldots, 2 n+1  \tag{40}\\
\left(r_{ \pm}^{i}\right)^{2}=\mp \operatorname{res}_{p_{ \pm i}} d \Omega_{ \pm} \tag{41}
\end{gather*}
$$

hold.

The pair of functions $\psi:=\left(\psi_{+}, \psi_{-}\right)$is the Baker-Akhiezer function on $\Gamma:=\left(\Gamma_{+} \bigsqcup \Gamma_{-}\right)$.

## Theorem

The Baker-Akhiezer function $\psi(z, \bar{z}, p)$ on $\Gamma$ satisfies the equation

$$
\left(\partial_{z} \partial_{\bar{z}}-u(z, \bar{z})\right) \psi(z, \bar{z}, p)=0
$$

with the potential $u=\partial_{\bar{z}} \xi_{1}^{+}=\partial_{z} \xi_{1}^{-}$
Moreover, the $2 n+1$-dimensional vector

$$
q=H y_{+}, \quad H \in S O(2 n+1, C)
$$

satisfies the equations:

$$
(q, q)=1, \quad\left(\partial_{z}^{i} q, q\right)=0, \quad i>0
$$

## The elliptic CM system

## Theorem

The Eqs. (30) are satisfied iff $\Gamma_{ \pm}$is the normalization of the spectral curve $\mathcal{C}_{N_{ \pm}}$of $N_{ \pm}$-particle elliptic Calogero-Moser (eCM) system.

Recall that the $N$-particle eCM is the Hamiltonian system on $\mathcal{X}_{N}=T^{*}\left(E^{N} \backslash \operatorname{diag}\right)=\left\{\left(p_{i}, z_{i}\right)_{i=1}^{N} \mid p_{i} \in \mathbb{C}, z_{i} \in E, z_{i} \neq z_{j}, i \neq j\right\}$, $E=\mathbb{C} / \mathbb{Z} \omega_{1} \oplus \mathbb{Z} \omega_{2}$, which is governed by the Hamiltonian

$$
\begin{equation*}
H_{2}=\frac{1}{2} \sum_{i=1}^{N} p_{i}^{2}+\nu^{2} \sum_{i<j} \wp\left(z_{i}-z_{j} ; \omega_{1}, \omega_{2}\right) \tag{42}
\end{equation*}
$$

This is an algebraic integrable system with the Lax operator

$$
\begin{equation*}
L(\alpha)=\left\|p_{i} \delta_{i j}+\nu \frac{\sigma\left(\alpha+z_{i}-z_{j}\right)}{\sigma\left(z_{i}-z_{j}\right) \sigma(\alpha)}\left(1-\delta_{i j}\right)\right\|_{i, j=1}^{N} \tag{43}
\end{equation*}
$$

whose flows linearized on the Jacobian of the spectral curve $C_{N} \subset M=\overline{\mathbb{C} \times(E \backslash\{\alpha=0\})}$

$$
\begin{equation*}
R(k, \alpha):=\operatorname{Det}(k-L(\alpha))=k^{N}+\sum_{j=1}^{N} k^{N-j} c_{j}(\alpha)=0 \tag{44}
\end{equation*}
$$

Explicitly

$$
R(k, \alpha)=f(k+\zeta(\alpha), \alpha)
$$

where

$$
\begin{equation*}
f(p, \alpha)=\frac{1}{\sigma(\alpha)} \sigma\left(\alpha+\frac{\partial}{\partial p}\right) H(p)=\frac{1}{\sigma(\alpha)} \sum_{n=0}^{N} \frac{1}{n!} \partial_{\alpha}^{n} \sigma(\alpha) \frac{\partial^{n} H}{\partial p^{n}} \tag{45}
\end{equation*}
$$

and $H$ is the monic degree $N$ polynomial

$$
\begin{equation*}
H(p)=p^{N}+\sum_{i=0}^{N-1} l_{i} p^{N-i} \tag{46}
\end{equation*}
$$

whose coefficients $I_{0}, \ldots, I_{N-1} \in \mathbb{C}$ are the integrals of motion of the the $N$-particle elliptic Calogero-Moser (eCM) system. Their values parametrize the eCM spectral curves.

## Turning points of the eCM system

## Theorem

A spectral curve of the eCM system admits a holomorphic involution $\sigma$ under which the marked point $P$ is fixed, i.e. $\sigma(P)=P$, if and only if it corresponds to a "turning point" of the CM system $\left(0, z_{i}\right)_{i=1}^{N}$.

In terms of $I_{i}$ the corresponding curves are those with $I_{\text {odd }}=0$.

## Example. O(3) model

Let $\Gamma$ be the spectral curve corresponding to a turning point of $N=2 \ell$-particle eCM system. For generic values of the parameters $I_{\text {even }}\left(\right.$ and $I_{\text {odd }}=0$ ) the curve $\Gamma$ is smooth of genus $g=2 \ell$ and has only two fixed points of the involution. Our construction of the solutions for the $O(3)$ sigma model requires curves with involution having at least 4 fixed points.
The equations

$$
\begin{equation*}
\partial_{\alpha} F(0,0)=0, \quad \partial_{p} F(0,0)=0 \tag{47}
\end{equation*}
$$

cut out an $(\ell-2)$ dimensional linear subspace in the space of parameters $I_{\text {even. }}$. The normalization of the singular spectral curve has genus $(2 \ell-3)$ and has 4 fixed points of involution. The corresponding quotient-curve is of genus $(\ell-2)$.

Introduce the function

$$
\begin{gather*}
\phi=\frac{\theta\left(A(p)-A\left(p_{3}\right)-Z\right) \theta\left(A\left(\gamma_{1}\right)+Z\right) \theta\left(A\left(\gamma_{2}\right)+Z\right)}{\theta\left(A\left(p_{3}\right)+Z\right) \theta\left(A(p)-A\left(\gamma_{1}\right)-Z\right) \theta\left(A(p)-A\left(\gamma_{2}\right)-Z\right)} \times \\
\times \frac{\theta\left(A(p)+\boldsymbol{A}\left(p_{3}\right)-A\left(\gamma_{1}\right)-A\left(\gamma_{2}\right)+z U-Z\right)}{\theta\left(A\left(p_{3}\right)-A\left(\gamma_{1}\right)-A\left(\gamma_{2}\right)+z U-Z\right)} \times \\
\times \exp \left(z \Omega_{2}(p)\right) \tag{48}
\end{gather*}
$$

Then

$$
r_{1} \phi\left(z, p_{1}\right)+i \phi\left(z, p_{2}\right)=w(z)=C \frac{\prod_{i=1}^{\ell} \sigma\left(z-a_{i}\right)}{\prod_{i=1}^{\ell} \sigma\left(z-b_{i}\right)}, \quad \sum_{i}\left(a_{i}-b_{i}\right)=0
$$

