On stratification of Hurwitz spaces of stable maps

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The most recent reference

M. Kazarian, S. Lando and D. Zvonkine, *Double Hurwitz Numbers and Multisingularity Loci in Genus* 0, IMRN 2021

Hurwitz spaces

Hurwitz spaces are spaces of meromorphic functions on complex algebraic curves, $\{f : C \to \mathbb{C}P^1\}$. To specify a Hurwitz space, one should specify

- the genus g of the curves;
- the degree d of the functions;
- the number of points of degenerate ramification in the target $\mathbb{C}P^1$ (all the other ramification points are presumed to be nondegenerate);
- the ramification profile (a partition of d) over each point of degenerate ramification.

Two functions $f_1 : C_1 \to \mathbb{C}P^1$ and $f_2 : C_2 \to \mathbb{C}P^1$ are considered as being the same if there is a biholomorphism $h : C_1 \to C_2$ such that $f_1 = f_2 \circ h$.

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Remark: If a Hurwitz space is nonempty (the above data is consistent), then the tuple of points of nondegenerate ramification in $\mathbb{C}P^1$ can serve as a set of local coordinates in it.

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- Geometry of Hurwitz spaces can be used to compute the Gromov–Witten invariants of the projective line $\mathbb{C}P^1$ (*Hurwitz numbers*).
- This can be used as a model example for computing Gromov–Witten invariants for arbitrary target varieties.

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- Cavalieri–Marcus (2014): compactification by rubber stable maps (for double Hurwitz spaces), . . .

Kontsevich's approach, in contrast to the other ones, leads to a *smooth* orbifold, and we use it to compactify Hurwitz spaces and study their geometry.

Simple Hurwitz spaces

A Hurwitz space is *simple* if it consists of functions with only one point of degenerate ramification, which we assume to be infinity. Its preimages are *poles*, and the ramification profile consists of the orders of the poles. The Hurwitz space $\mathcal{H}_{g;\kappa}$ of degree K meromorphic functions on genus g curves with poles of orders $\kappa = (k_1, \dots, k_n) \vdash K$ is a connected orbifold of dimension $n + K + 2g - 2$. Denote by $\overline{\mathcal{H}}_{g;\kappa}$ the irreducible component in the space of stable maps containing $\mathcal{H}_{g;\kappa}$.

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The Hurwitz space $\overline{\mathcal{H}}_{g;\kappa}$ is fibered over the moduli space $\overline{\mathcal{M}}_{g;n}$ of stable genus g curves with $n = \ell(\kappa)$ marked points (which are the poles of the functions).

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- the ELSV formula (2000) for connected simple Hurwitz numbers

$$h_{g;\kappa} \sim \int_{\overline{\mathcal{M}}_{g;n}} \frac{c(\Lambda^\vee)}{\prod_{i=1}^n (1 - k_i\psi_i)} = \int_{\overline{\mathcal{M}}_{g;n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{\prod_{i=1}^n (1 - k_i\psi_i)};$$

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  as a consequence, $h_{g;\kappa}$ is a polynomial in the parts of κ.
- the generating function for linear Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g;n}} \lambda_i \psi_1^{k_1}\dots\psi_n^{k_n}$$

  is a 1-parameter family of solutions to the KP (Kazarian, 2009).

Double Hurwitz spaces

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- There is no analogue of the ELSV formula that expresses connected double Hurwitz numbers in terms of intersection indices on moduli spaces of curves;
- However, they are known to be piecewise polynomial in the parts of the two partitions, which suggests an existence of such a formula; several formulas of this kind have been conjectured, but the integral in all of them is over certain conjectural spaces.

Double ramification cycles

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Definition of double ramification cycles in $\overline{\mathcal{M}}_{g;n}$ is more subtle and requires considering virtual cohomology classes. There are expressions of these cycles in terms of ψ- and κ-classes (conjectured by Pixton and proved by Janda, Pandharipande, Pixton and Zvonkine in 2017).

Strategy for studying double Hurwitz numbers

It is more convenient to consider simple Hurwitz spaces with additional marked points on the source curves (that eventually become zeroes). Let $\mathcal{H}$ be the simple Hurwitz space of meromorphic functions on genus g curves with ramification profile $\kappa \vdash K$ over infinity, $\mathcal{H} = \mathcal{H}_{g;r|\kappa}$. The double Hurwitz space $\mathcal{X}_{(\lambda)}$ is a stratum in $\mathcal{H}_{g;r|\kappa}$, the closure of the locus of functions having zeroes of orders $l_1, \dots, l_r$, $l_i \geq 0$ at the additional marked points. The codimension of the stratum is $|\lambda| = l_1 + \dots + l_r$.

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$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "P"'] & & Y \arrow[dl, "Q"] \\ & P\mathcal{H} & \end{tikzcd} \tag{1}$$

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X — the universal source curve over $P\mathcal{H}$ with poles removed;
Y — the universal target curve over $P\mathcal{H}$ with infinity removed;
F — the universal mapping whose restriction to a fiber of P over a point $f \in \mathcal{H}$ coincides with f.

Thom's universality principle

For a triple like

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "P"'] & & Y \arrow[dl, "Q"] \\ & P\mathcal{H} & \end{tikzcd} \qquad (2)$$

a relative version of *Thom's universality principle for global singularities* can be applied. It asserts that
for a generic mapping F, the cohomology class Poincaré dual to the subvariety in X consisting of points where the restriction of F to a fiber of P acquires a singularity of a given type admits a universal expression in terms of the relative Chern class

$$c(F) = \frac{c(F^*TY)}{c(TX)} = 1 + c_1(F) + c_2(F) + \dots.$$

Singularity types

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- For a node of a fiber, the possible singularity types are $I_{k,l}$, which are A_{k-1} when restricted to one branch and A_{l-1} when restricted to the other. The corresponding versal deformations are spaces of trigonometric polynomials (rational functions with two poles of given order).

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- For a node of a fiber, the possible singularity types are $I_{k,l}$, which are A_{k-1} when restricted to one branch and A_{l-1} when restricted to the other. The corresponding versal deformations are spaces of trigonometric polynomials (rational functions with two poles of given order).
- Also unavoidable are nonisolated singularities, where f takes an irreducible component of a curve to a single point. The corresponding versal deformations are simple Hurwitz spaces.

Kazarian's principle

Kazarian's principle extends Thom's principle to the case of multisingularities. A *multisingularity* of a given tuple of types, is a point in Y whose preimages under F are singularities of these types. A double Hurwitz space $\mathcal{X}_{(\lambda)}$ is the projection under Q of the locus $\Sigma_{l_1-1,l_2-1,\dots,l_r-1}[Y] \subset Y$ of multisingularities $(A_{l_1-1}, A_{l_2-1}, \dots, A_{l_r-1})$.

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$$\sum [\Sigma_{l_1-1,l_2-1,\dots,l_r-1}[Y]] t_{i_1} t_{l_2} \dots$$

for the cohomology classes dual to the loci of multisingularities in Y is the exponent of the generating function

$$\sum_{i_1,\dots,i_m} F_*(R_{i_1,\dots,i_m}) t_{i_1} t_{i_2} \dots t_{i_m}.$$

Generating function for A_k multisingularities

In the case when all the fibers are smooth, we have

$$c(F) = \frac{c(F^*TY)}{c(TX)} = \frac{1-\psi}{1+\xi}.$$

Here ψ and ξ are, respectively, the first Chern class of the relative *cotangent* line bundle to the source and the relative *tangent* line bundle to the target curve. All Chern classes of F can be expressed from just two classes, ψ and ξ.

Generating function for A_k multisingularities

Define the *rescaled one-part Schur polynomials* $s_k = s_k(t_1, t_2, \dots)$ by

$$e^{-\frac{1}{\psi}(t_1x+t_2x^2+t_3x^3+\dots)} = s_0 + s_1x + s_2x^2 + \dots$$

Theorem

The generating function $\mathcal{R}$ of residual polynomials for the map F is

$$\exp(-\frac{\xi}{\psi}\mathcal{R}) = 1 + \xi s_1 + \xi(\xi+\psi)s_2 + \xi(\xi+\psi)(\xi+2\psi)s_3 \dots .$$

It is a solution to the scaled KP hierarchy of partial differential equations. In particular, it solves the first scaled KP equation of the form

$$\frac{\partial^2\mathcal{R}}{\partial t_2^2} = 2\psi\xi\left(\frac{\partial^2\mathcal{R}}{\partial t_1^2}\right)^2 + \frac{4}{3}\frac{\partial^2\mathcal{R}}{\partial t_1\partial t_3} - \frac{1}{3}\psi^2\frac{\partial^4\mathcal{R}}{\partial t_1^4}.$$

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For general case, this generating function must be corrected. For $g=0$, correction terms are known up to codimension 5.

Generating function for double Hurwitz numbers in genus 0

Any subvariety in $P\mathcal{H}$ admits a *degree*, which is the intersection index of the class $\frac{1}{1-\xi}$, $\xi = \mathcal{O}(1)$. The degree of the double singularity space is, essentially, the double Hurwitz number $h_{g;\kappa,\lambda}$. For $g = 0$ and a given λ, we collect the double Hurwitz numbers into a generating function

$$h_{(\lambda)} = \sum_{\kappa} h_{\kappa,(\lambda)} q_{k_1} q_{k_2} \dots .$$

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Theorem

The generating functions $h_{(\lambda)}$ are polynomials in the generating functions

$$z_{d,r}(q) = \sum_{K,n} \frac{1}{n!} \binom{n+r-3}{d} K^{n+r-3-d} \sum_{k_1+\dots+k_n=K} \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} q_{k_i}.$$

Recursion

The proof of the Theorem is effective and based on a recursion for a certain generating function for the enhanced functions $h_{(\lambda)}$. The enhancement is achieved by assigning powers of ψ-classes to the additional marked points.

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Several explicit formulas:

$$
\begin{aligned}
h_{(2)} &= z_{0,1} + z_{1,1}, \\
h_{(3)} &= z_{0,1} - \frac{z_{0,1}^2}{2} + 3\,z_{1,1} + 2\,z_{2,1}, \\
h_{(4)} &= z_{0,1} - \frac{5\,z_{0,1}^2}{2} + 6\,z_{1,1} - 2\,z_{0,1}\,z_{1,1} + 11\,z_{2,1} + 6\,z_{3,1}, \\
h_{(2,2)} &= -6\,z_{0,1} + z_{0,1}^2 + z_{0,2} - 11\,z_{1,1} + 2\,z_{1,2} - 6\,z_{2,1} + 2\,z_{2,2}, \\
h_{(2,2,2)} &= 85\,z_{0,1} - 40\,z_{0,1}^2 - 18\,z_{0,2} + 6\,z_{0,1}\,z_{0,2} \\
&\quad + z_{0,3} + 225\,z_{1,1} - 24\,z_{0,1}\,z_{1,1} - 51\,z_{1,2} + 6\,z_{0,1}\,z_{1,2} + 3\,z_{1,3} \\
&\quad + 274\,z_{2,1} - 84\,z_{2,2} + 6\,z_{2,3} + 120\,z_{3,1} - 54\,z_{3,2} + 6\,z_{3,3}.
\end{aligned}
$$

Thank you
for your attention