On the rationality of Fano 3-folds over non-closed fields
(based on a joint work with A. Kuznetsov)

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Fano = smooth Fano 3-fold over

$k \neq \bar{k}$, char $k = 0$.

Invariants X Fano

$\rho(X)$ Picard number

$\mathrm{Pic}(X) = \mathbb{Z}^{\rho}$

$\rho(\bar{X})$ $\bar{X} = X \otimes \bar{k}$.

$\mathrm{Pic}(X) \subset \mathrm{Pic}(\bar{X})^{G} \subset \mathrm{Pic}(\bar{X})$

$G = \mathrm{Gal}$

If $X(k) = \emptyset \Rightarrow$ =

$-K_X$ index (Fano)

$i(X) = \max\{ i \mid -K_X = iH \}$

$-K_X = iH \leftarrow$

Degree $d(X) = H^3$] $i(X) > 1$

Genus $g(X) = \frac{1}{2}(-K_X)^3 + 1$] $i(X) = 1.$

Fact X Fano 3-fold

$\Rightarrow i(X) \leq 4$

$i(X) = 4$ $X = \mathbb{P}^3$

$i(X) = 3 \Rightarrow X = Q \subset \mathbb{P}^4.$

$i(X) = 1$ or 2. (our assumption)

$i(X) = 2 \overset{\text{def}}{\Rightarrow}$ del Pezzo 3-fold.

$i(X) = 1 \overset{\text{def}}{\Rightarrow}$ prime Fano

$k = \bar{k}$ $\quad \rho(X) = 1.$

15 irred. families Fanos.

del Pezzo case $\Rightarrow d(X) = 1, \dots, 5$

$i(X) = 1$ prime Fano 3-folds $\Rightarrow g = 2, \dots, 10, 12.$

Rationality

$d(X) = 1, \dots, 3$ del Pezzo $\Rightarrow X$ not rational

$g(X) = 2, \dots, 6, 8, \Rightarrow X$ not rational

$d(X) = 4, 5$

$g(X) = 7, 9, 10, 12$

only these Fanos are interesting for consideration

X rational.

Remark $g(X) = 6$ and 4, known general member of family not rational.

$\boxed{k \neq \bar{k}}$ $\quad \rho(\bar{X}) = 1.$

$\left.\begin{array}{l} d = 5 \\ \text{del Pezzo} \end{array}\right] \Longrightarrow X$ rational.

$\boxed{g = 7, 12 \text{ prime}} \Longrightarrow X$ rational

$\Updownarrow$

$X(k) \neq \emptyset$

Theorem. $g = 10$ $\quad X$ rational
• also Tschinkel & Hassett •

$\Longleftrightarrow X(k) \neq \emptyset$ & $F_2(X)(k) \neq \emptyset$

family of conics

Remark. $F_2(\bar{X})$ abelian surface.

$g = 9$ $\quad X$ rational $\Longleftrightarrow$

$F_3(X)(k) \neq \emptyset$

family of $p_a = 0$ curves of degree $= 3$

Remark $\quad F_3(\bar{X})$ abelian 3 fold.

$\underline{\underline{d=4}}$ Benoist & Wittenberg
Tschinkel Hassett.

X $\quad X$ rat $\Longleftrightarrow$ $F_1(X)(k) = \emptyset$

$\exists$ line over k.

$\Rightarrow$ based on techniques of
· Benoist
· Wittenberg.

Fano 3-folds with
$\boxed{\rho(X)=1,}$ $\rho(\bar{X}) > 1$.

$\operatorname{Pic}(\bar{X}) = \mathbb{Z}^{\rho} \curvearrowleft G$

2-dim analog del Pezzo surfaces
$d = 1, \ldots, 9$

3-dim case $\rho(\bar{X}) > 1$
Mori - Mukai 88 families

Example $\mathbb{P}^3 \longleftarrow Bl_{pt}\, \mathbb{P}^3$

$\bar{X}$ G-Fano G finite group
$G \curvearrowright \operatorname{Pic}(\bar{X})$ $\overset{K}{\cdot}$

G-Fano 3-folds with $\rho > 1$ over $\mathbb{k} = \bar{\mathbb{k}}$ [P- 2013]

		G-del Pezzo 3-folds with $\rho > 1$		
$\rho(X)$	H^3	X	$h^{1,2}$	Rat? / $\bar{\mathbb{k}}$
3	6	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	0	+ / $\mathbb{k}$
2	6	$W_6 \subset \mathbb{P}^2 \times \mathbb{P}^2$, divisor of bidegree $(1,1)$	0	+

		G-Fano 3-folds with $\iota(X) = 1$ and $\rho > 1$		
$\rho(X)$	$g(X)$	X	$h^{1,2}$	Rat? / $\bar{\mathbb{k}}$
2	7	a) $X \subset \mathbb{P}^2 \times \mathbb{P}^2$, divisor of bidegree $(2,2)$ b) $X \xrightarrow{2:1} W_6$, branch divisor $\in \lvert -K_{W_6} \rvert$	9	−
3	7	$X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, branch divisor $\in \lvert -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1} \rvert$	8	−
2	11	$D_1 \cap D_2 \cap D_3 \subset \mathbb{P}^3 \times \mathbb{P}^3$, D_i is of bidegree $(1,1)$	3	+
4	13	$X_{(1,1,1,1)} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	1	+
2	15	blowup of $Q \subset \mathbb{P}^4$ along a twisted quartic curve	0	+
3	16	$D_{(0,1,1)} \cap D_{(1,0,1)} \cap D_{(1,1,0)} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	0	+

$\mathbb{k}'$

$\mathbb{P}^3 \leftarrow X \rightarrow \mathbb{P}^3$

Assumption $X(\mathbb{k}) \neq \emptyset$.

$x \in X$ $\mathbb{k}$-point.

Sarkisov links

Theorem 1 *Let $X = X_6$ be a del Pezzo 3-fold with $\rho(X) = 1$ and $\rho(X_{\bar{\Bbbk}}) > 1$. Suppose that X has a $\Bbbk$-point x. Then there exists the following Sarkisov link*

$$\begin{tikzcd}
& \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr] \arrow[rr, dashed, "\textit{flop}"] & & \tilde{X}^+ \arrow[dl] \arrow[dr, "\varphi"] & \\
X & & X_0 & & X^+
\end{tikzcd}$$

has a $\Bbbk$-point

where σ is the blowup of x and φ is an extremal Mori contraction. Moreover:

- *$X_{\bar{\Bbbk}} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \implies X^+ \simeq \mathbb{P}^3$ and φ is the blowup of three conjugate points.*

- *$X_{\bar{\Bbbk}} \simeq W_6 \implies X^+ = X_2^+ \subset \mathbb{P}^3$ is a quadric and φ is a $\mathbb{P}^1$-bundle.*

In particular, X is $\Bbbk$-rational.

Theorem 2 *Let $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ be a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$ and $\rho(X_{\bar{\Bbbk}}) > 1$. Suppose that X has a $\Bbbk$-point x which does not lie on a line. Then there exists the following Sarkisov link*

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr] \arrow[rr, dashed, "flop"] & & \tilde{X}^+ \arrow[dl] \arrow[dr, "\varphi"] & \\ X & & X_0 & & X^+ \end{tikzcd}$$

where σ is the blowup of x and φ is an extremal Mori contraction. Moreover:

+
- *$g = 16 \implies X^+_{\bar{\Bbbk}} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and φ is the blowup of a smooth rational curve of degree 6. In particular, X is $\Bbbk$-rational.*

$\rho(\bar{X}^+) = 1.$

+
- *$g = 15 \implies X^+ = X^+_5 \subset \mathbb{P}^6$ is a smooth del Pezzo 3-fold of degree 5 and φ is the blowup of a disjoint union of two conics. In particular, X is $\Bbbk$-rational.*

?
- *$g = 13 \implies X^+ = X^+_3 \subset \mathbb{P}^4$ is a 4-nodal cubic 3-fold and φ is the blowup of singular points.*

?
- *$g = 11 \implies X^+ \simeq \mathbb{P}^2$ and φ is a conic bundle with discriminant curve of degree 4.*

$\implies$ Corollary X $\Bbbk$-unirational!

$g = 11$

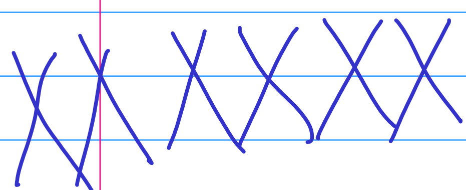
$X \underset{\text{bir}}{\sim} X^+ = Y$

$\rho(\bar{X}) = 2$ $\bar{k}/k$ $\quad f: Y \longrightarrow \mathbb{P}^2 \supset \Delta$

standard conic bundle

$\rho(\bar{Y}) = 3$ over $\bar{k}$ not minimal.

Δ discriminant $\quad f^{-1}(\Delta)$ splits over $\bar{k}$



$\tilde{\Delta}$ étale.

$\downarrow$

Δ smooth

Th (Benoist, Wittenberg)

X/S k-minimal conic bundle $\quad S$ rat surface

Δ smooth connected

(a) $\tilde{\Delta}/\Delta$ splits over k'/k quadratic extension

(b) X/S admits rat. section over k'/k

(c) Δ non-hyperelliptic

$\Longrightarrow X$ is not k-rational

Corollary. X $g=11$ not rational.

$\underline{\underline{g=13}}$ $\quad \overline{X} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
$(1,1,1,1)$

$\overline{X} \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \qquad h^{1,2}(X)=1.$

blowup of ell. curve

X k-unirational.

G = image of $Gal(\bar{k}/k)$
in $Aut(Pic(\overline{X}))$
$\qquad\qquad\qquad \| $
$\qquad\qquad\qquad \mathbb{Z}^4.$

$\underline{G \subset S_4}$ ① $S_4, A_4, D_4, \mu_2 \times \mu_2$
transitive
② μ_4

$G \supset \mu_2 \times \mu_2$

① $G = \mu_2 \times \mu_2$

Degeneration techiques

(Shinder, Nicaise).

stably rationality "open" conditions

$X \xrightarrow[\text{degen.}]{} X'$ not stably rat. $\Longrightarrow$ — | — | — very general member

$X' = \{x_0 y_0 t_0 z_0 = x_1 y_1 t_1 z_1\}$

$\subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

toric, singular, O.D.P's 6

$R_{k'/k} \mathbb{G}_m^1$ k'/k degree 4 extension

Weil ext.

$\to$ $\left(N: \quad R_{k'/k} \mathbb{G}_m^1 \longrightarrow \mathbb{G}_m^1\right.$

$R^{(1)}_{k'/k} \mathbb{G}_m \subset X'$ open.

∘ Known to be non-stably rat.

$H^1(G, \mathrm{Pic}(\tilde{X}'))$ — smooth model.