Nik: K3 ⇝ smooth K3

more symmetric

higher g

?

Here: K3 ⇝ nonnormal surface

mirror symmetry

Degenerations of elliptic K3 surfaces

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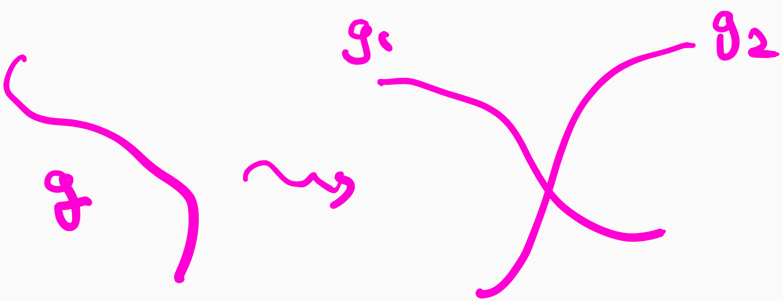
Steklov Institute (Moscow), Nikulin-70 Conference

University of Georgia

Cf: $\overline{M}_g$



$g = g_1 + g_2$

Sources

This talk is based on:

Sketched/proved 1 Made a conj. for 2

- Brunyate, *A modular compactification of the space of ellip- tic K3 surfaces*, Ph.D. thesis, University of Georgia, 2015

/I

- * Alexeev, Brunyate, Engel, *Compactifications of moduli of elliptic K3 surfaces: stable pair and toroidal*, arXiv:2002.07127 (2019): Improved 1
  Proved the conj 2

Main goals:

$F_{ell} = \Gamma \backslash \Omega^{18}(\mathrm{II}_{2,18})$ q. proj

1. Construct a geometrically meaningful compactification $\overline{F}_{\mathrm{el}}$ of the moduli space of elliptic surfaces
2. Anwer: Is $\overline{F}_{\mathrm{el}}$ toroidal? What is the fan?

Related work:

Add more fibers w. weights, usually dim $M = 18 + n$

- Ascher, Bejleri, *Compact moduli of elliptic K3 surfaces*, arXiv:1902.10686 (2019)
- Odaka, *PL density invariant for type II degenerating K3 surfaces, Moduli compactification and hyperKahler metrics*, arXiv:2010.00416 (2020)

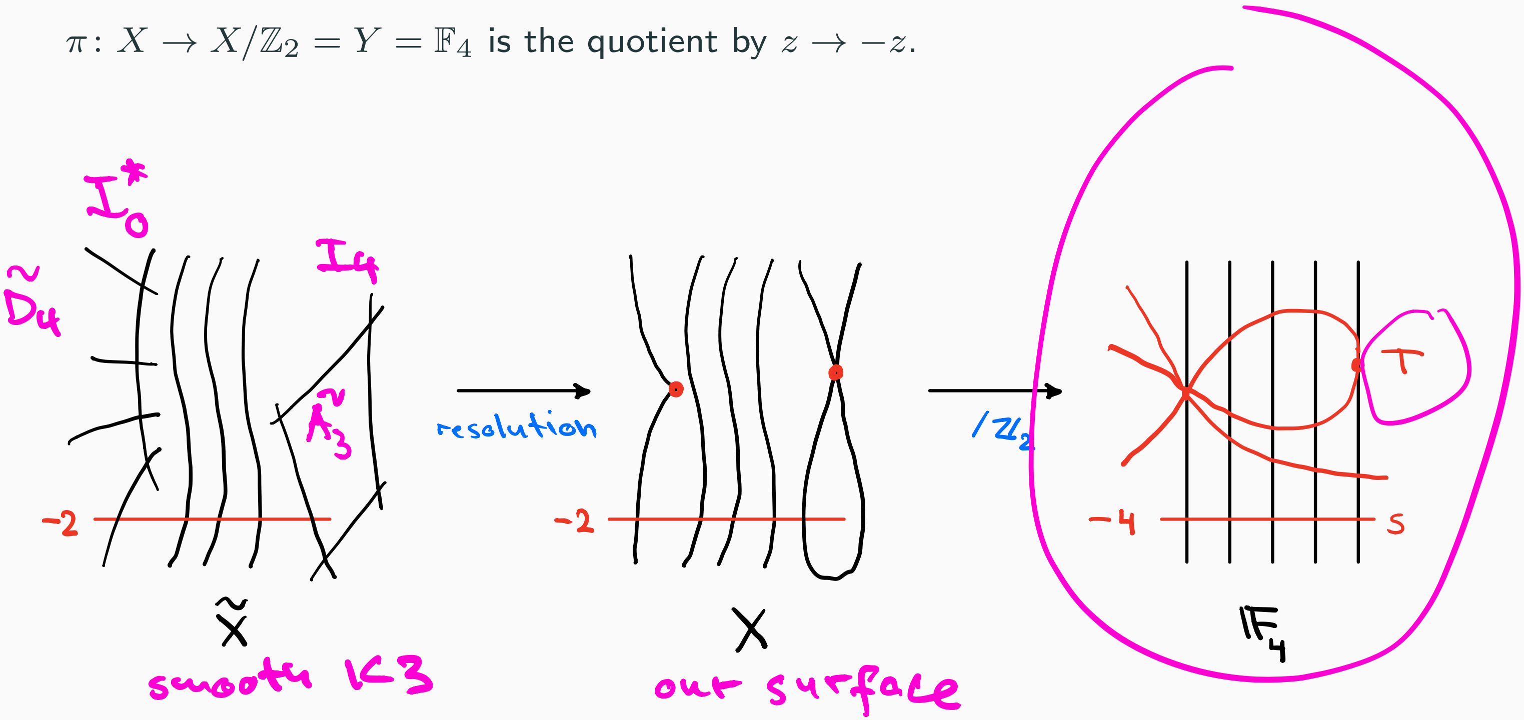
Extended 1 to $\mathbb{Z}[\frac{1}{6}]$. + diff. geometry

Elliptic K3 surfaces

Here, elliptic K3 surfaces are in a Weierstrass form: $X \to \mathbb{P}^1_x$ with a section s and with irreducible fibers.

$$z^2 = y^3 + A_8(x)y + B_{12}(x), \qquad \Delta_{24}(x) = 4A^3 + 24B^3$$

$\pi\colon X \to X/\mathbb{Z}_2 = Y = \mathbb{F}_4$ is the quotient by $z \to -z$.

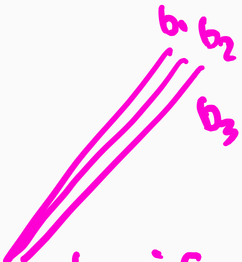


$0 < \epsilon << 1$

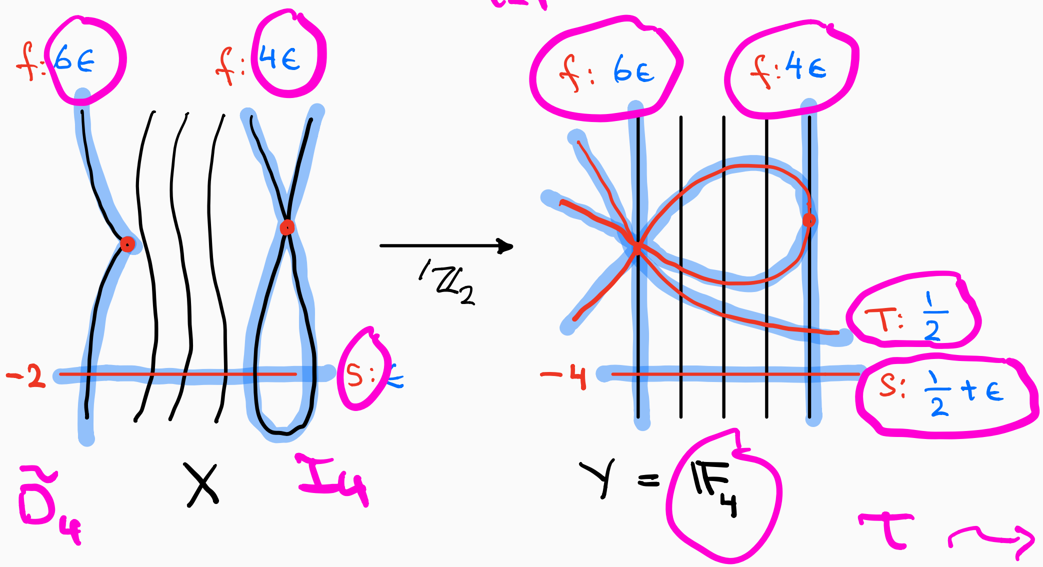
Elliptic K3 pair is $(X, \epsilon(s + \sum_{i=1}^{24} F_i))$, where F_i are the singular fibers, $0 < \epsilon << 1$.

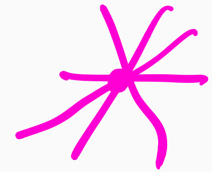
$$\left(X, \epsilon(s + \sum_{i=1}^{24} F_i)\right) \text{ is lc } \iff \left(Y, (1/2 + \epsilon)\, s + \frac{1}{2} T + \epsilon \sum f_i\right) \text{ is lc}$$

$\bigcup_{i=1}^{24} F_i = (\Delta_{24}(x))$



lc if $\sum b_i \leq 1$





lc if $\sum b_i \leq 2$

lc $\frac{1}{2} + \frac{1}{2} = 1 \leq 1$

Stable slc elliptic K3 pairs

Definition

A stable elliptic K3 pair is a stable slc (KSBA) limit of K3 pairs $(X, \epsilon(s + \sum_{i=1}^{24} F_i))_t$.

Theorem (K-SB-A et al, 1988-2020)

Any 1-parameter family of elliptic K3 pairs has a unique stable limit.
There exists a projective moduli space $\overline{F}_{\mathrm{el}}^{\mathrm{slc}} \supset F_{\mathrm{el}}$ *of stable slc elliptic K3 pairs.*

Main Question

Can one describe all stable limits? Is $\overline{F}_{\mathrm{el}}^{\mathrm{slc}}$ a toroidal compactification of $F_{\mathrm{el}} = \Omega/\Gamma$?

Main Theorem (Alexeev-Brunyate-Engel, 2020)

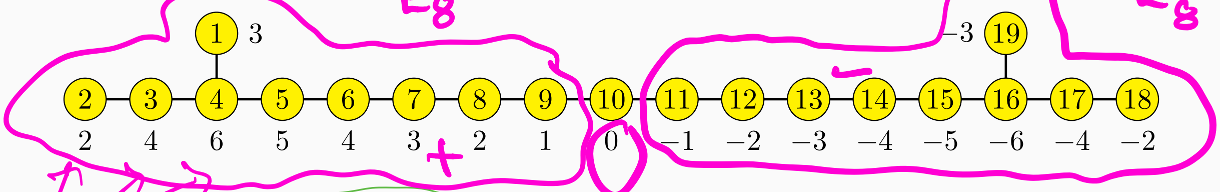
There is a complete classification of all stable limits in terms of Dynkin diagrams.
The normalization of $\overline{F}_{\mathrm{el}}^{\mathrm{slc}}$ is toroidal compactification for an explicit fan $\mathfrak{F}$.

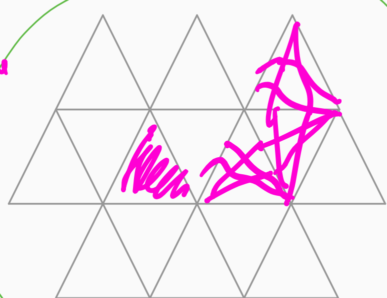
Let $II_{2,18} = H^{\perp}$ in $H^2(\mathrm{K3}, \mathbb{Z})$ and $\Omega = \{w \in II_{2,18} \otimes \mathbb{C} \mid w^2 = 0, w\bar{w} > 0\}$.

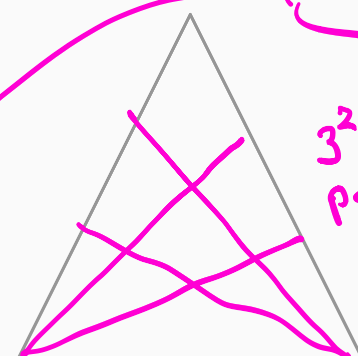
Then $F_{\mathrm{el}} = \Omega / O_+(II_{2,18})$. A toroidal compactifications $\overline{F}^{\mathfrak{F}}$ is defined by an $O_+(II_{1,17})$-equivariant fan $\mathfrak{F}$ in $II_{1,17} \otimes \mathbb{R}$ with support on $\mathcal{C}_{\mathbb{Q}}$, where $\mathcal{C} = \{v \mid v^2 > 0, vh > 0\}$.

One has $O_+(II_{1,17}) = S_2 \ltimes W$, and the Weyl group W is defined by

$\tilde{E}_8$ $\tilde{E}_8$





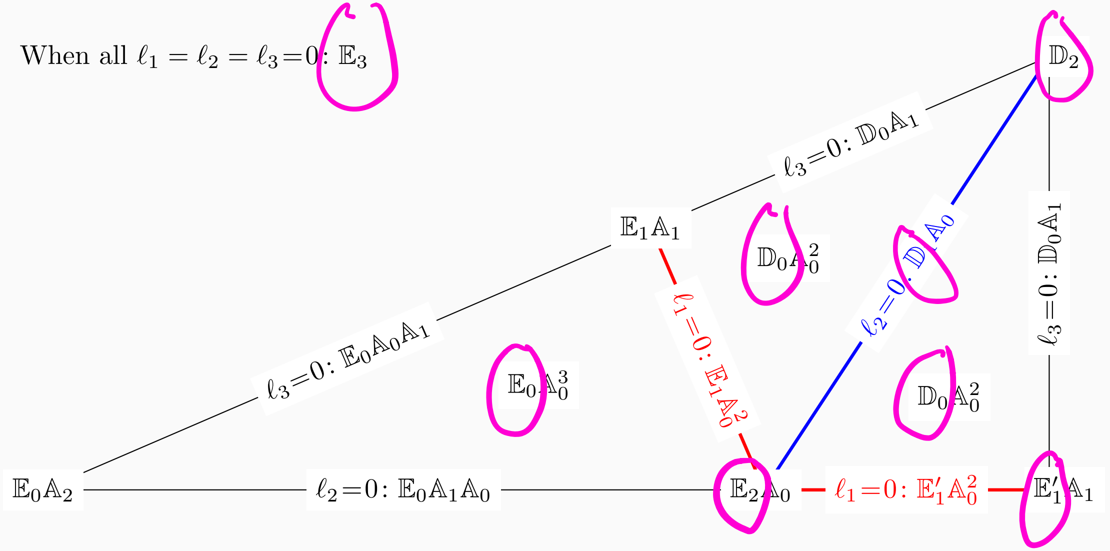


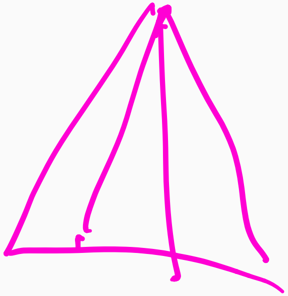
3^2 parts

$\overline{\mathcal{C}}_{\mathbb{Q}}$

$\{v^2>0\} = \mathcal{C}$

When all $\ell_1 = \ell_2 = \ell_3 = 0$: $\mathbb{E}_3$



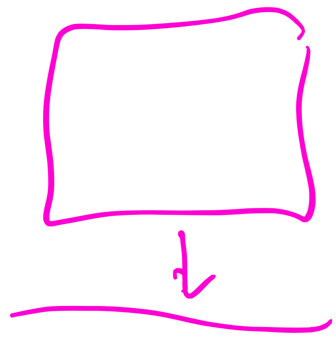


Main idea:

smaller ell. surfaces

Q: What are smallest ell. surf?

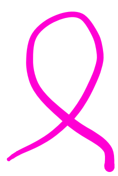
① Rational :



$\Delta_{12}(x)$

$\mathbb{P}^1$

②



$\times \mathbb{P}^1$

$\downarrow \quad \leftarrow$

$\mathbb{P}^1$

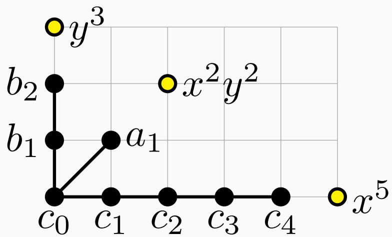
K3 :	$\Delta_{24} = \Delta_{12\cdot 2}$		2
ratul :	Δ_{12} :	$\Delta_{12\cdot 1}$	1
	Δ_0 :	$\Delta_{12\cdot 0}$	0

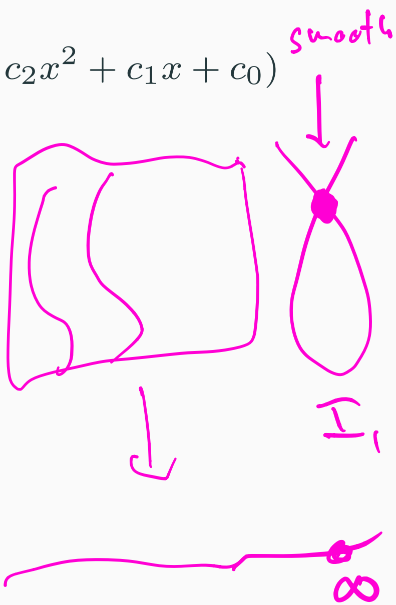
The E_8 family = rational elliptic surfaces with I_1 fiber at $x = \infty$

$$\color{red}{xyz} = (\color{red}{z^2} + a_1 z) + (\color{red}{y^3} + b_2 y^2 + b_1 y) + (\color{red}{x^5} + c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0)$$

Substituting $w = z + \frac{1}{2}(a_1 + xy)$, this gives the rational elliptic surface

$$w^2 = (y^3 + b_2 y^2 + b_1 y) - \frac{1}{4}(xy - a_1)^2 + (x^5 + c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0)$$

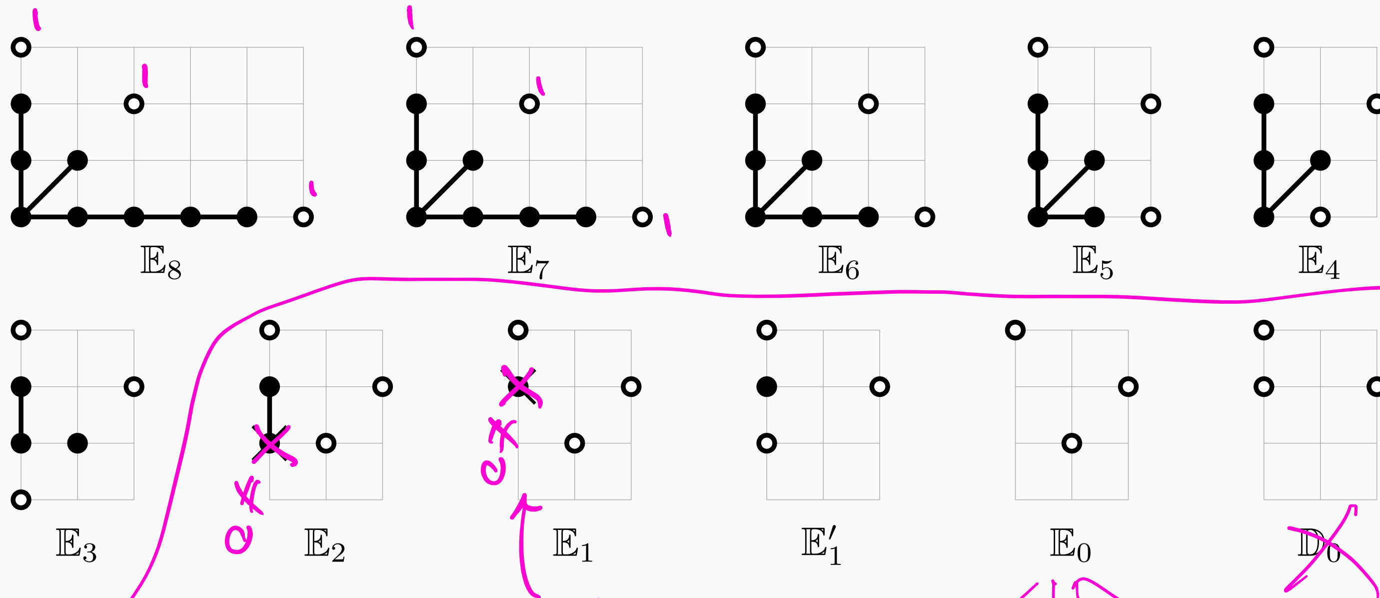




Theorem (classical?, A-Thompson'19, A-Brunyate-Engel'20)
This is a universal family of rational elliptic surfaces with I_1 fiber at $x = \infty$. The base is $\mathbb{C}^8 = \mathrm{Hom}(E_8, \mathbb{C}^*)/W(E_8)$. Setting $a_i = b_i = c_i = 0$, gives the E_8 singularity.

$(\mathbb{C}^*)^8/W(E_8)$

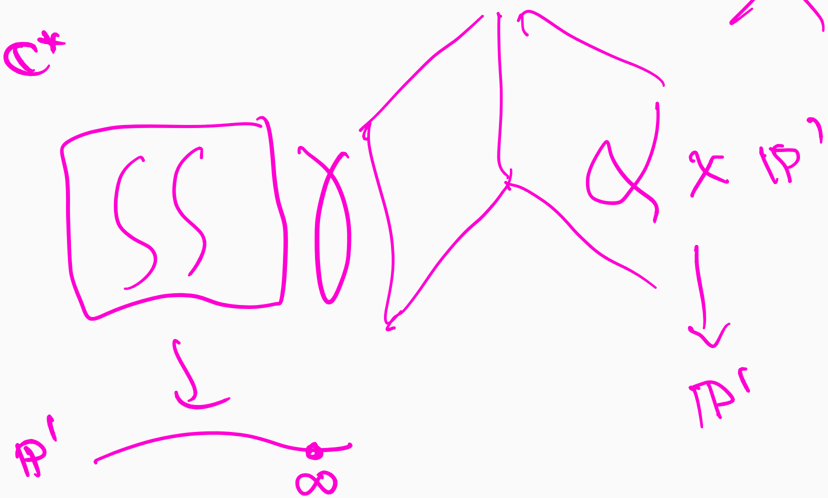
All E_n families = rational elliptic surfaces with I_{9-n} fiber at $x = \infty$



$\mathbb{C}^*$

super sfd

$\mathbb{C}^m$ / finite group

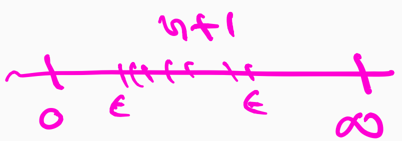


All ADE elliptic families

$dP = Bl_k \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$

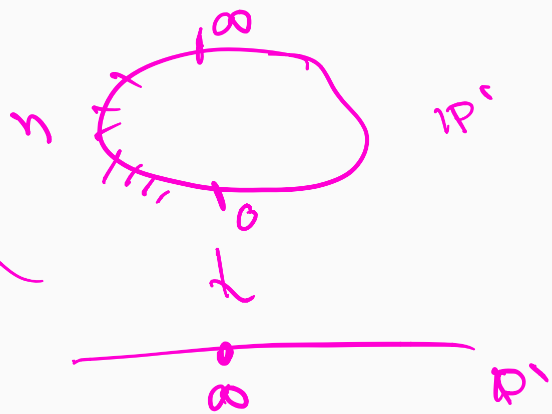
$Bl_8 \mathbb{P}^2$ $\quad$ $Bl_1 \mathbb{P}^2$ $\quad$ $\mathbb{P}^2$ $\quad$ $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{tikzcd}
 & & & & & & & E_1' \arrow[ddr] & \\
E_8 \arrow[r] \arrow[dr] & E_7 \arrow[r] \arrow[dr] & E_6 \arrow[r] \arrow[dr] & E_5 \arrow[r] \arrow[dr] & E_4 \arrow[r] \arrow[dr] & E_3 \arrow[r] \arrow[dr] & E_2 \arrow[r] \arrow[dr] \arrow[ur] & E_1 \arrow[r] \arrow[dr] & E_0 \\
D_8 \arrow[r] \arrow[dr] & D_7 \arrow[r] \arrow[dr] & D_6 \arrow[r] \arrow[dr] & D_5 \arrow[r] \arrow[dr] & D_4 \arrow[r] \arrow[dr] & D_3 \arrow[r] \arrow[dr] & D_2 \arrow[r] \arrow[dr] & D_1 \arrow[r] \arrow[dr] & D_0 \\
A_8 \arrow[r] & A_7 \arrow[r] & A_6 \arrow[r] & A_5 \arrow[r] & A_4 \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1 \arrow[r] & A_0
\end{tikzcd}$$

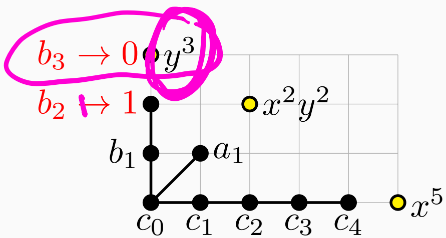


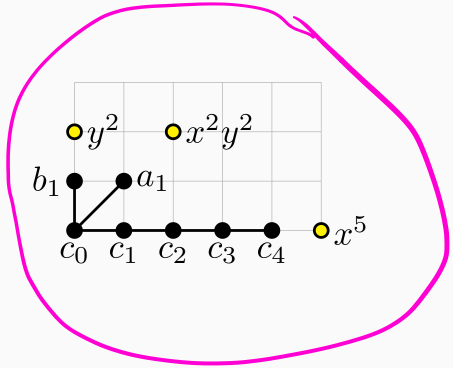
Losev–Manin curves

$L_{n+3} \leftarrow \overline{M}_{0,n+3}$



$E_8 \to D_7$ and the D_n families

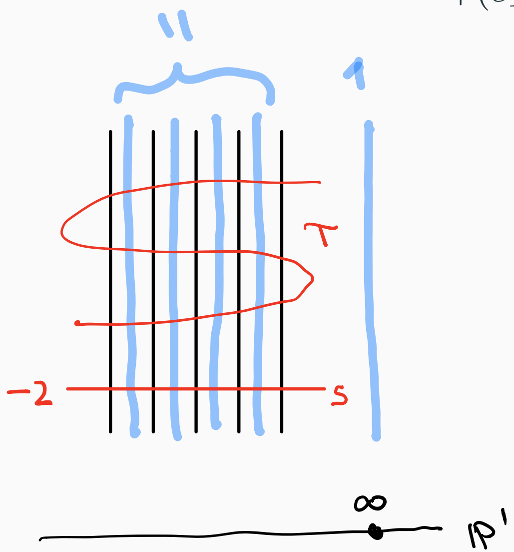


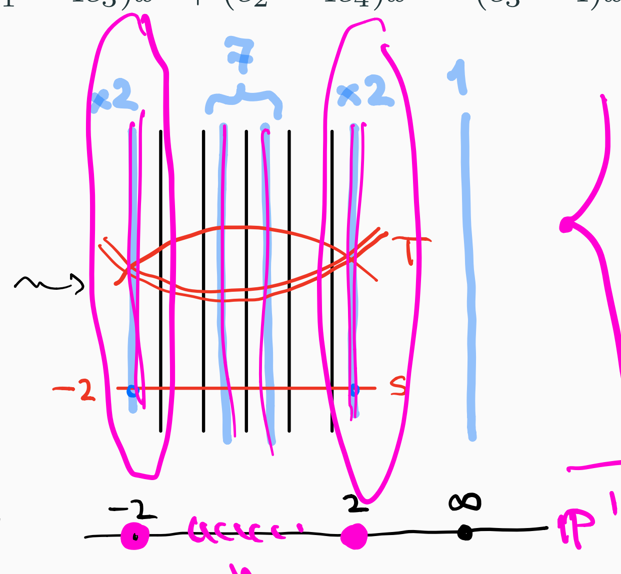


$\Delta(x) \equiv 0$

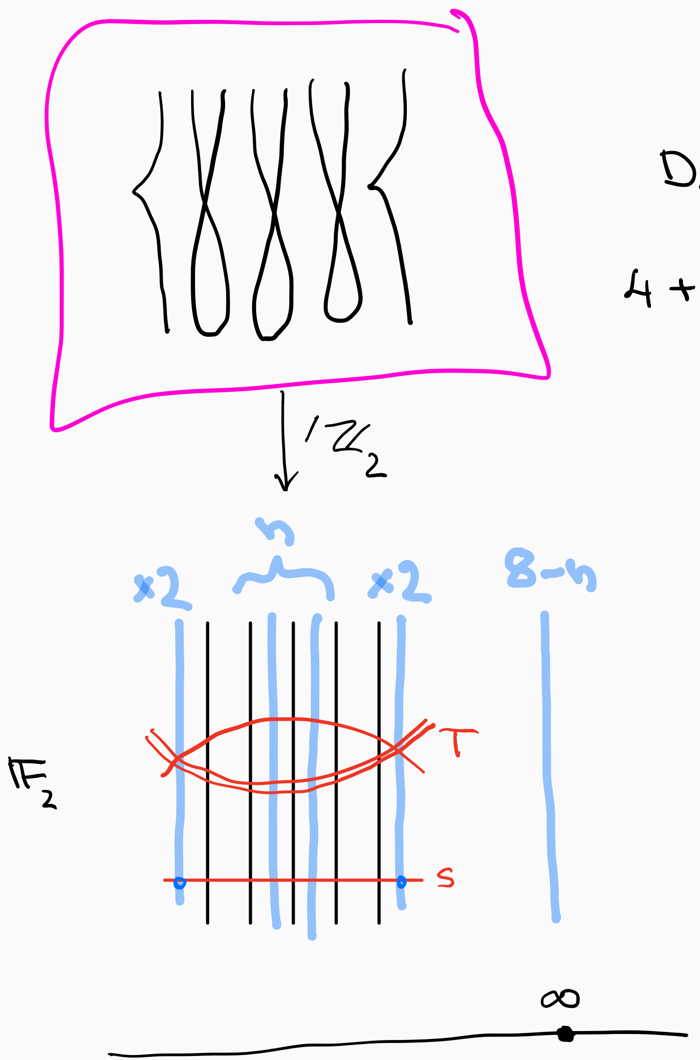
$\lim F_i = ?$

$$\lim_{b_3 \to 0} \frac{\Delta(x)}{b_3^2} = (x-2)^2(x+2)^2 \times \big(a_1^2 + b_1^2 - 4c_0 + (a_1b_1 - 4c_1)x + (c_0 - 4c_2)x^2 + \\ +(c_1 - 4c_3)x^3 + (c_2 - 4c_4)x^4 - (c_3 - 4)x^5 + c_4x^6 + x^7\big)$$



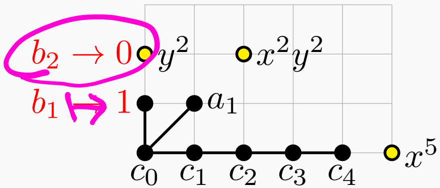


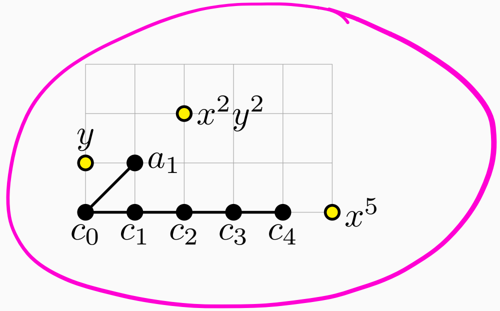
extra fibers



D_n $(n \geq 0)$

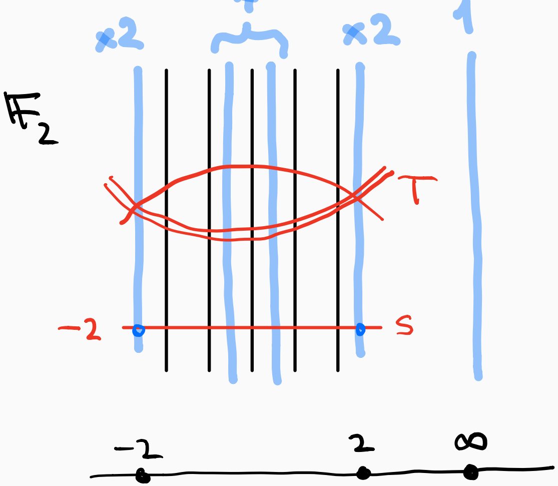
$4 + n$ fibers

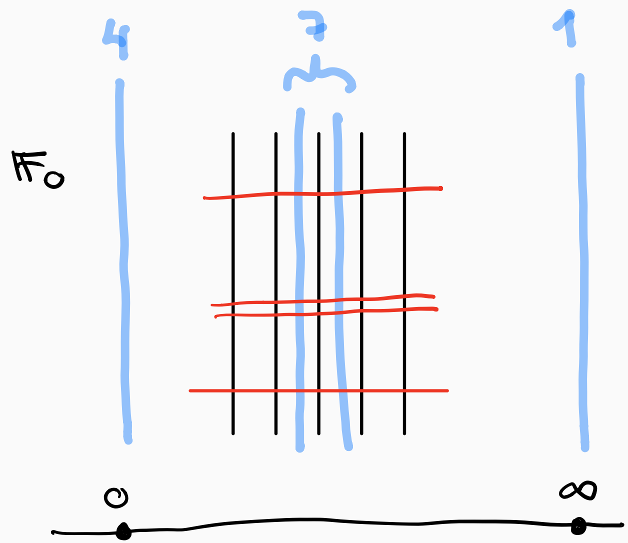




$\Delta(x) \equiv 0$

$$\lim_{b_3, b_2 \to 0} \frac{\Delta(x)}{b_3^2} = x^4(1 + a_1 x + c_0 x^2 + c_1 x^3 + c_2 x^4 + c_3 x^5 + c_4 x^5 + x^7)$$

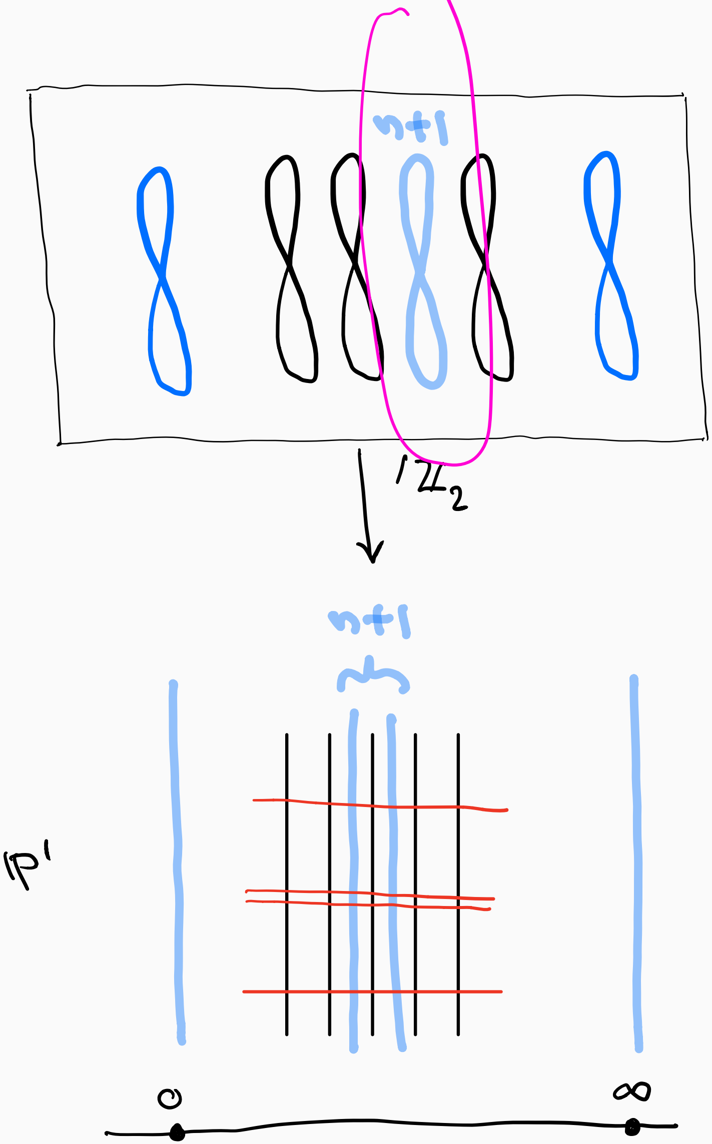




X

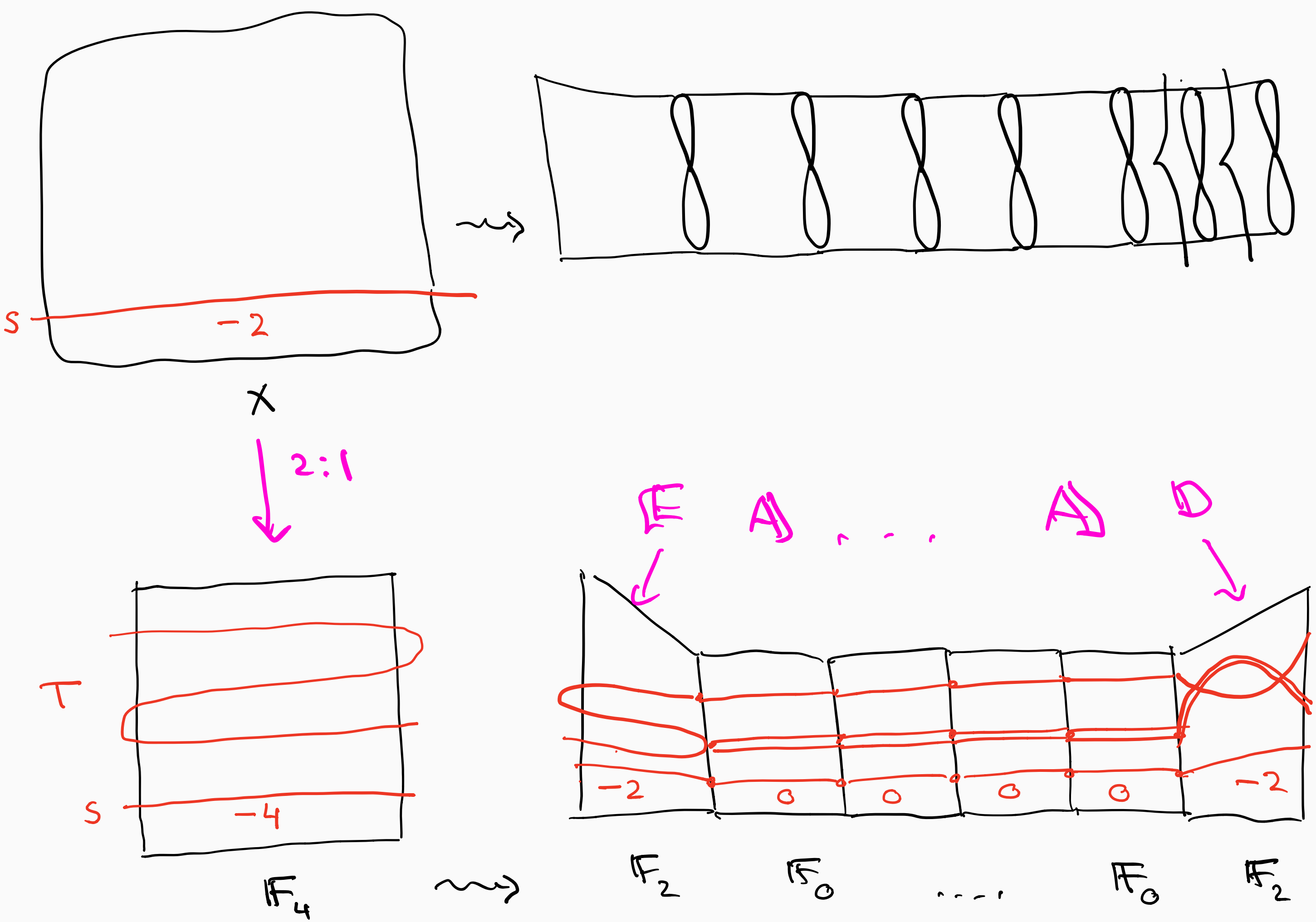
$A_n \ (n \geq 0)$

$n+1$ fibers over $\mathbb{C}^*$

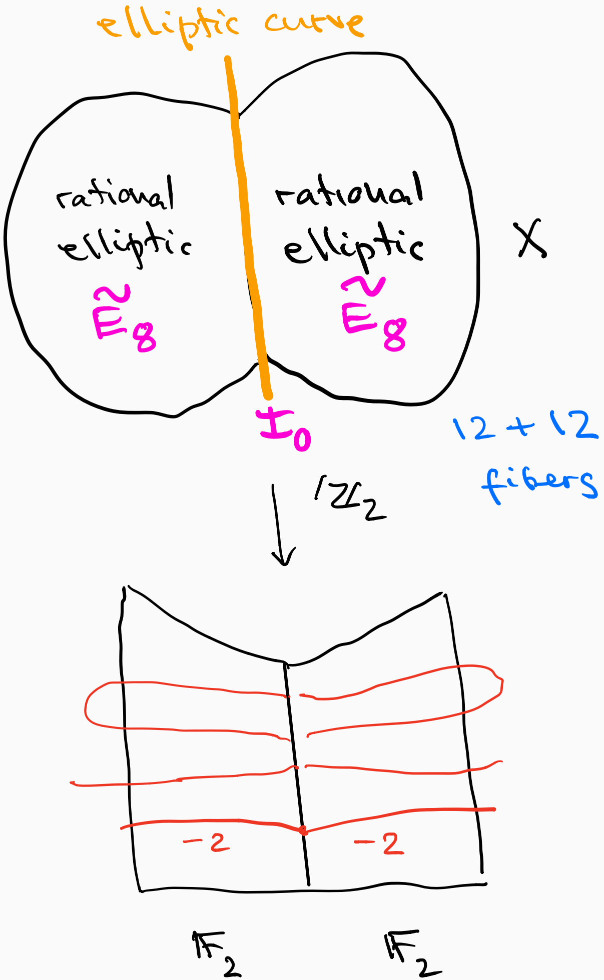


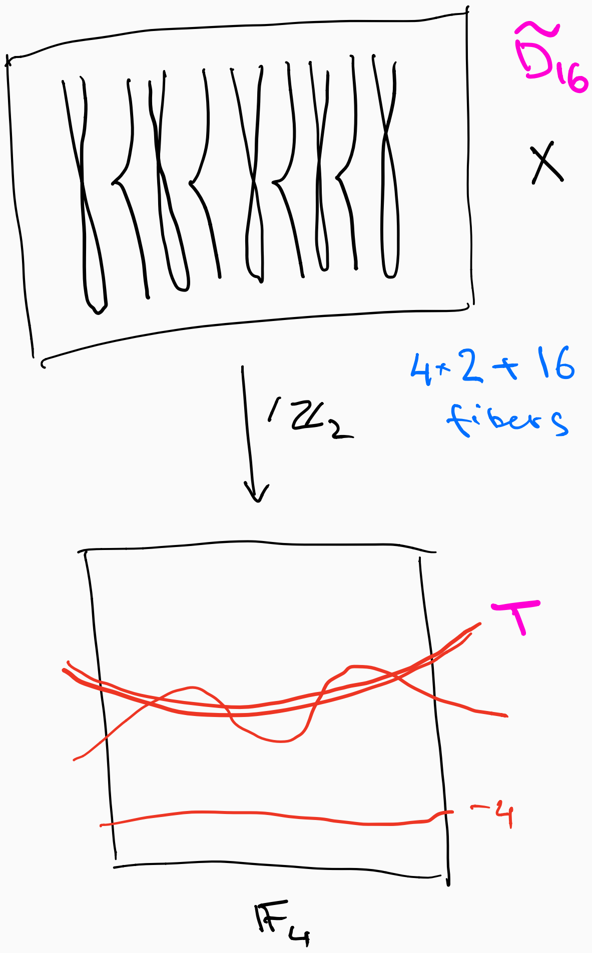
$Y = \mathbb{P}^1 \times \mathbb{P}^1$

All type III degenerations: $(E/D)A\ldots A(E/D)$



Type II degenerations: $\widetilde{E}_8\widetilde{E}_8$ and $\widetilde{D}_{16}$





Proof of the Main Thm

Step 1. For each monodromy invariant $\lambda \in \overline{C}_{\mathbb{Q}} \subset II_{1,17} \otimes \mathbb{R}$, $\lambda^2 \geq 0$, we construct:

divisor mo

1. A family of "enhanced" Kulikov models $(\widetilde{X}, \epsilon\widetilde{D})$ with big and nef $\widetilde{D} = \widetilde{s} + \sum \widetilde{f}_i$.
2. A family of stable models $(X, \epsilon D)$ with ample $D = s + \sum f_i$.

$|n\widetilde{D}|$

Step 2. These are families over blowups of $\overline{F}^{\mathfrak{F}}_{\mathrm{el}}$. We use these families to prove that $\phi\colon \overline{F}^{\mathfrak{F}}_{\mathrm{el}} \dashrightarrow \overline{F}^{\mathrm{slc}}_{\mathrm{el}}$ is regular. We then show that ϕ is the normalization map.

Step 3. The stable pairs constructed are exactly the $(E/D)A \ldots A(E/D)$, $\widetilde{E}_8\widetilde{E}_8$, $\widetilde{D}_{16}$ pairs described above $\implies$ all of them appear and there are no others.

$X_t \rightsquigarrow X_0$

$\downarrow$

$C \ni 0$

$X_0 \longrightarrow \overline{X}_0$

slc pair

$(\overline{X}_0, \epsilon\overline{D}_0)$

Main technology: divisor models (enhanced Kulikov models)