Enriques surfaces and Leech lattice

S. Kondō

23 Oct. 2020 Nikulin's 70

§1. Enriques surfaces with $|\mathrm{Aut}(X)| < \infty$

§2. Leech lattice

§3. Coble surfaces with $|\mathrm{Aut}(Y)| < \infty$

MATHEMATISCHES INSTITUT
UNIVERSITÄT ERLANGEN-NÜRNBERG
Prof. Dr. Wolf Barth

(1)

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Postanschrift - mailing address:
Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstrasse 1½
D-8520 Erlangen
Tel. (09131) 85 2455 (Durchwahl)

Herrn
Prof. Y. Namikawa
Max-Planck-Institut f. Mathematik
Gottfried-Claren-Str. 26

5300 Bonn 3

Sent to	C. Burns	Y. Namikawa
	M. Cossec	C. Peters
	I. Dolgachev	M. Reid
	E. Looijenga	J. Shah
	S. Mukai	A. Verra

Dear Namikawa!

I am writing to inform you about Nikulin's results on automorphisms of Enriques surfaces. Let Y be an Enriques surface and X the K3-surface such that $Y = X/\{1,\theta\}$ where θ is the involution of X without fixed points. Let S be the Picard lattice of X and

$$S_\pm = \{x \in S \mid \theta(x) = \pm x\}$$

(1) $$\Delta_\pm = \{\delta_\pm \in S_\pm \mid \delta_\pm^2 = -4,\ \exists \delta_\mp \in S_\mp = -4 \ \text{and}\ (\delta_\pm + \delta_\mp)/2 \in S\}$$

$[\Delta_-] \subset S_-$ the sublattice generated by Δ_-

$$K = [\Delta_-]\ (\tfrac{1}{2})$$

Then K is a root lattice ($K = \oplus A_m,\ D_n,\ E_k$).

Let $\xi : K \bmod 2 \to S_+ \bmod 2$
$\qquad \delta_- \bmod 2 \to \delta_+ \bmod 2$

the homomorphism defined by (1) and $H = \operatorname{Ker} \xi$ the isotropic subgroup of form K mod2. The pair (K,H) is called the root invariant of the Enriques surface Y.

Theorem 1. The root invariant (K,H) defines Aut(Y) up to finite groups.

Theorem 2. There are exactly 6 possibilities for the root invariant of an Enriques surface with a finite group of automorphisms:

$(E_8 \oplus A_1, \{\bar{0}\})$, $(D_9, \{\bar{0}\})$, $(D_5 \oplus D_5, \{\bar{0}\}$,

$(E_6 \oplus A_4, \{\bar{0}\})$, $(A_9 \oplus A_1, \mathbb{Z}/2\mathbb{Z})$, $(E_7 \oplus A_2 \oplus A_1, \mathbb{Z}/2\mathbb{Z})$

The example of Dolgachev corresponds to the root invariant $(E_8 \oplus A_1, \{\bar{0}\})$. The example of Barth-Peters ($\mathrm{Aut}Y \sim \mathbb{Z}$) corresponds to the root invariant $(E_8, \{\bar{0}\})$.

The proof of theorem 1 uses the global Torelli theorem for K3-surfaces, theorem 2 - elliptic pencils on Enriques surfaces. Nikulin wants to note that these results are generalized to K3-surfaces with a transcendental involution and its factors.

Yours sincerely

Wolf Barth

(W. Barth)

Докл. Акад. Наук СССР
Том 277 (1984), № 6

Soviet Math. Dokl.
Vol. 30 (1984), No. 1

3

ON A DESCRIPTION OF THE AUTOMORPHISM GROUPS OF ENRIQUES SURFACES

UDC 512.774

V. V. NIKULIN

1. Some definitions. By a *lattice* we understand a nondegenerate symmetric bilinear form over $\mathbf{Z}$. A lattice S is *even* if $x^2 = x \cdot x$ is even for any $x \in S$. The symbol $S(r)$, where $r \in Q$, denotes the multiplication of the form S by r. Furthermore, A_n, D_m and E_k are indecomposable negative definite lattices generated by their elements of square (-2) which form the root systems $\mathbf{A}_n, \mathbf{D}_m$ and $\mathbf{E}_k$ respectively; $U = \left\langle \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right\rangle$. The symbol $\oplus$ denotes the direct sum.

By an *R-invariant* (or *root invariant*) we understand a pair (K, H), where K is a negative definite lattice generated by its elements of square (-2) (i.e., the direct sum of A_n, D_m and E_k), and $H \subset K/2K$ is the isotropy subgroup with respect to the quadratic form $K \bmod 2$ (i.e., $x^2/2 \bmod 2$, where $x \in K$), satisfying the following condition: the overlattice $K_H = [K; x/2$, where $x \in K$ and $x \bmod 2 \in H]$ of the lattice K, which is of finite index over K, does not contain elements of square (-1).

By an *embedding* of root invariants $\varphi\colon (K, H) \to (K', H')$ we understand an embedding of the lattices $K \subset K'$ for which $H = \overline{\varphi}^{-1}(\overline{\varphi}(K/2K) \cap H')$, where $\overline{\varphi} = \varphi \bmod 2$. Root invariants are considered up to isomorphism given by this definition.

By an *r-invariant* (modulo 2 root invariant or modulo 2 root system) we understand a finite set $\overline{\Delta}$ of a quadratic space f over the two-element field, which satisfies the following conditions:

a) $f(\overline{\delta}) = \overline{1}$ for all $\overline{\delta} \in \overline{\Delta}$.

b) $s_{\overline{\delta}}(\overline{\Delta}) = \overline{\Delta}$ for all $\overline{\delta} \in \overline{\Delta}$, where $s_{\overline{\delta}}(x) = x + f(x, \overline{\delta})\overline{\delta}$ is a reflection.

An embedding $\overline{\Delta}_1 \subset \overline{\Delta}_2$ of r-invariants and an isomorphism between them are easily defined.

Every R-invariant defines an r-invariant $r = \overline{R} = \overline{(K, H)}$ equal to $\overline{\Delta(K)} \bmod H$, where

(1) $$\Delta(K) = \{\delta \in K \mid \delta^2 = -2\}, \qquad \overline{\Delta(K)} = \Delta(K) \mod 2.$$

Note that an embedding of R-invariants naturally defines an embedding of their corresponding r-invariants.

By an *r-invariant* $\overline{\Delta}$ *of type* $\mathcal{E}_8$ we understand an invariant such that there exists a negative definite sublattice $K \subset E_8$ of finite index generated by $\Delta(K)$, and $\overline{\Delta} \simeq \Delta(K) \bmod 2E_8 \subset \overline{\Delta(E_8)}$ (see (1)). In other words, $\overline{\Delta} \simeq \overline{(K, H)}$ for some R-invariant (K, H) for which $\operatorname{rk} K = 8$ and there exists an embedding of R-invariants $(K, H) \subset (E_8, \{\overline{0}\})$.

2. Root invariants and the groups of automorphisms of Enriques surfaces. Let Y be an Enriques surface (over $\mathbf{C}$), i.e., $Y = X/\{1, \theta\}$, where X is a $K3$ surface with a fixed-point-free involution θ (recall that X is simply connected). Let S_+ and S_- be the fixed and the antifixed parts of the action of θ on the Picard lattice S of X. It is well known and easy to check that $S_+(2^{-1}) \simeq U \oplus E_8$ and S_- is negative

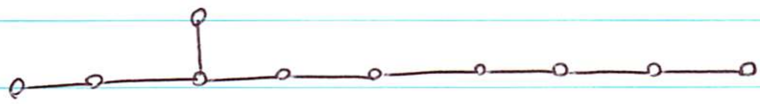
1980 *Mathematics Subject Classification.* Primary 14J25, 14L30.

$/k = \bar{k}$, $\mathrm{char}\, k = p \geq 0$

S : an <u>Enriques surface</u> $\iff$ $K_S \equiv 0$, $B_2(S) = 10$

If $p \neq 2$, then $\exists\, X \xrightarrow{2:1} S$ étale s.t. X is a K3 surface.

$\mathrm{Num}(S) := NS(S)/\text{torsions} \cong E_{10}$ even unimodular lattice of sign $(1,9)$.



$\rho : \mathrm{Aut}(S) \longrightarrow O(E_{10})$ $\quad |\mathrm{Ker}\, \rho| < +\infty$.

<u>Difference between K3 and Enriques</u>

- K3 $\quad \delta \in \mathrm{Pic}(X)$, $\delta^2 \geq -2 \Rightarrow \delta \geq 0$ or $-\delta \geq 0$
- Enriques. $\quad \delta^2 = -2$ $\not\Rightarrow$ 〃
  false.

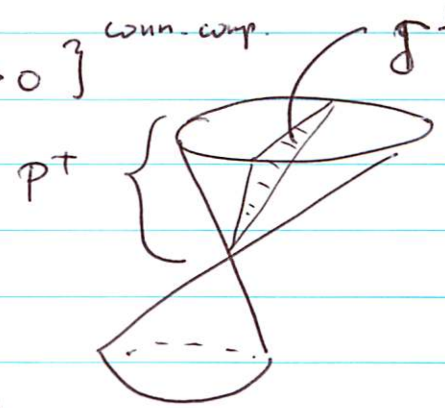
A gen. Enriques has no (-2)-curves.
$(\mathbb{P}^1)$.

$$O(E) \supset W(S) := \left\langle \left\{ s_\delta \;\middle|\; s_\delta : x \longrightarrow x + \langle x, \delta \rangle \delta \;,\; \delta = [\mathbb{P}^1] \right\} \right\rangle$$

$\circlearrowright$

$$P^+(E_{10}) := \{ x \in E_{10} \otimes \mathbb{R} \mid x^2 > 0 \}^{\text{conn. comp.}}$$

$$C(S) := \left\{ x \in P^+(E_{10}) \;\middle|\; \langle x, \delta \rangle > 0 \;\; \forall \delta = [\mathbb{P}^1] \right\}.$$



<u>ample cone</u>.
〃
a fundamental domain of $W(S)$.

$O(E) \cong \{\pm 1\} \cdot W(S) \cdot \mathrm{Aut}(C(S))$.

$\mathrm{Im}(\rho : \mathrm{Aut}(S) \to O(E)) \cap W(S) = \{1\}$.

$\Rightarrow$. $[O(E_{10}) : W(S)] < \infty \implies |\mathrm{Aut}(S)| < \infty$.

($\Leftarrow$ holds $/\mathbb{C}$).

- Enriques : $|\mathrm{Aut}(S)| = +\infty$ if S is generic.
- G. Fano (1910) : an example of S with $|\mathrm{Aut}(S)| < \infty$. (Type VII)
- I. Dolgachev (1984) : " (Type I).
- V. Nikulin (1984) : R-invariants and classification
- S. K— (1986) : Construction and "
- G. Martin (2019) : char $p \neq 2$.
- T. Katsura, G. Martin, K— (2020) : char $p = 2$

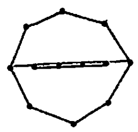
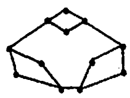
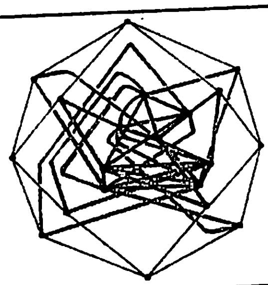
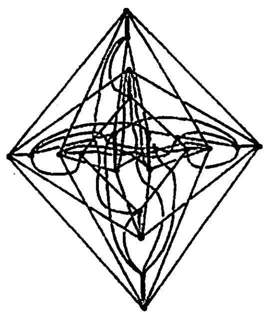
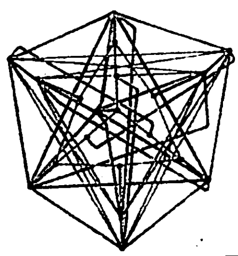
$|\mathrm{Aut}(S)| < \infty \Rightarrow$ Any elliptic fibration on S is <u>extremal</u>
(i.e. "Mordell-Weil group" is finite.)

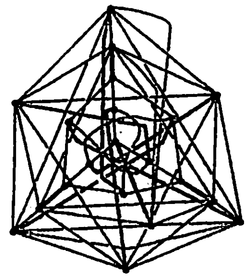
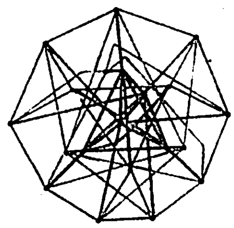
$\Leftrightarrow [O(E_{10}) : W(S)] < \infty$.

$\uparrow$

E. Vinberg's theorem.

Classification table in char $p \neq 2$ or μ_2-surface in char 2.

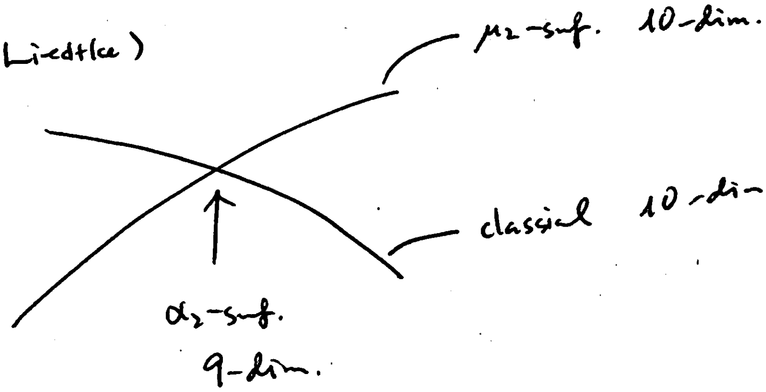
Type	Dual Graph of (−2)-curves	Aut	Aut_{nt}	Char(k)	Moduli
I		D_8	$\mathbb{Z}/2\mathbb{Z}$	any	$\mathbb{A}^1 - \{0, -256\}$
II		$\mathfrak{S}_4$	$\{1\}$	any	$\mathbb{A}^1 - \{0, -64\}$
III		$(\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_8$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2$	unique
IV		$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$	$\{1\}$	$\neq 2$	unique
V		$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2, 3$	unique

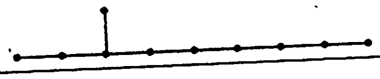
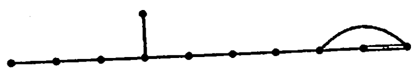
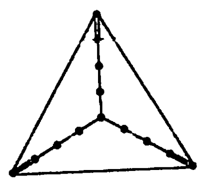
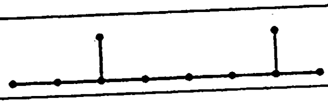
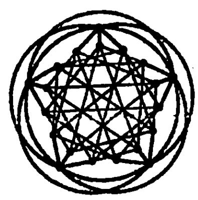
VI		$\mathfrak{S}_5$	$\{1\}$	$\neq 3,5$	unique
VII		$\mathfrak{S}_5$	$\{1\}$	$\neq 2,5$	unique

$\underline{p=2}$ Bombieri–Mumford. Pic^{τ}_{S} can. covering

Enriques surface $=$
- classical $\mathbb{Z}/2\mathbb{Z}$ μ_2-cover (inseparable)
- μ_2-surface μ_2 $\mathbb{Z}/2\mathbb{Z}$-cover (separable)
- α_2-surface α_2 α_2-cover (inseparable)

moduli (Liedtke)



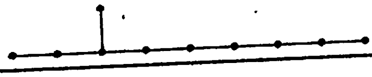
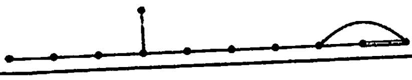
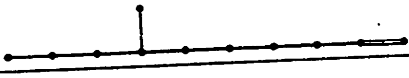
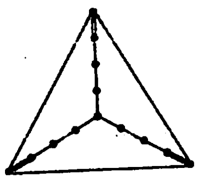
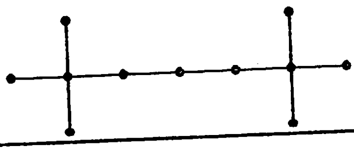
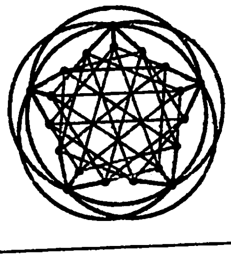
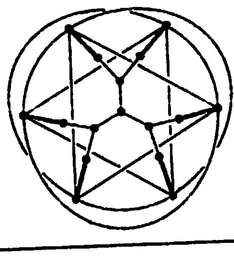
Type	Dual Graph of (-2)-curves	$\mathrm{Aut}(X)$	$\mathrm{Aut}_{ct}(X)$	dim
$\tilde{E}_8$		$\mathbb{Z}/11\mathbb{Z}$	$\mathbb{Z}/11\mathbb{Z}$	0
${\tilde{E}_7}^2$		$\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/14\mathbb{Z}$	$\{1\}$ or $\mathbb{Z}/7\mathbb{Z}$	1 or 0
$\tilde{E}_6 + \tilde{A}_2$		$\mathbb{Z}/5\mathbb{Z} \rtimes \mathfrak{S}_3$	$\mathbb{Z}/5\mathbb{Z}$	0
$\tilde{D}_8$		Q_8	Q_8	1
VII		$\mathfrak{S}_5$	$\{1\}$	0

α_2-surface

Difference between $p=2$, classical, α_2-surf and others:

· A multiple fiber of elliptic fibration is of additive type (= not $\tilde{A}_n$-type).

(9)

Type	Dual Graph of (-2)-curves	$\mathrm{Aut}(X)$	Aut_{nt}	dim
$\tilde{E}_8$		$\{1\}$	$\{1\}$	1
$\tilde{E}_7{}^2$		$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	2
$\tilde{E}_7{}^1$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1
$\tilde{E}_6 + \tilde{A}_2$		$\mathfrak{S}_3$	$\{1\}$	0
$\tilde{D}_8$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1
$\tilde{D}_4 + \tilde{D}_4$		$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	2
VII		$\mathfrak{S}_5$	$\{1\}$	1
VIII		$\mathfrak{S}_4$	$\{1\}$	1

Cossec-Dolgachev (extra-special)

Ekedahl, Shepherd-Barron, Salomonsson. (exceptional)

classical

<u>Example 1</u> : Hessian quartic of a cubic surface.

$$R : \sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \lambda_i x_i^3 = 0 \subset \mathbb{P}^4 \quad \text{(Sylvester form)}$$

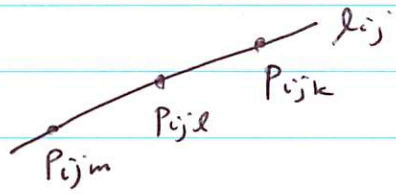
$\downarrow$ Hessian : $\det\left(\frac{\partial^2}{\partial z_i \partial z_j} f_3(z_1, \cdots, z_4)\right) = 0$

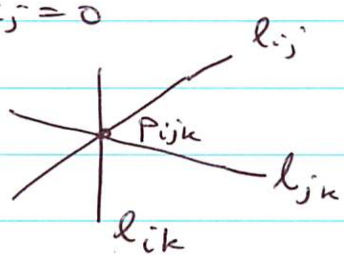
$$H : \sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \frac{1}{\lambda_i x_i} = 0 \subset \mathbb{P}^4. \longleftarrow X \quad \text{mini. resol.}$$

K3.

$$\sigma : (x_i) \longrightarrow \left(\frac{1}{\lambda_i x_i}\right)$$

$$\begin{cases} H \text{ has } 10 \text{ nodes } P_{ijk} : x_i = x_j = x_k = 0 \\ \quad '' \quad 10 \text{ lines } \ell_{ij} : x_i = x_j = 0 \end{cases}$$

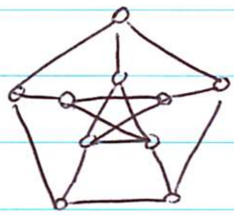




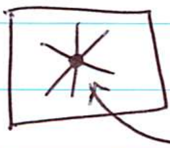
$S := X/\langle\sigma\rangle$ an Enriques surface $\qquad \sigma : P_{ijk} \leftrightarrow \ell_{mn}$

$\cup$

$10\ \mathbb{P}^1$

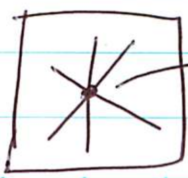
$\Gamma =$ 

Petersen graph. $\mathrm{Aut}(\Gamma) \cong \mathfrak{S}_5$

$\mathbb{P}^3 \supset$ a cubic surface  hyperplane section

<u>Eckardt point</u>

In case $R : \sum x_i = \sum \lambda_i x_i^3 = 0$

If $\lambda_i = \lambda_j$, then $x_i + x_j = 0$ is  P_{mnl}

$\{x_i + x_j = 0\} \cap \text{Hessian} = 2\ell_{ij} \overset{\exists}{+} 2 \text{ lines}$

$\circlearrowright \sigma$

$\Rightarrow$ We obtain a new $\mathbb{P}^1$.

- If $\lambda_1 = \cdots = \lambda_5$, R is called "Clebsch diagonal cubic".

  S contains additionally 10 $\mathbb{P}^1$. al.

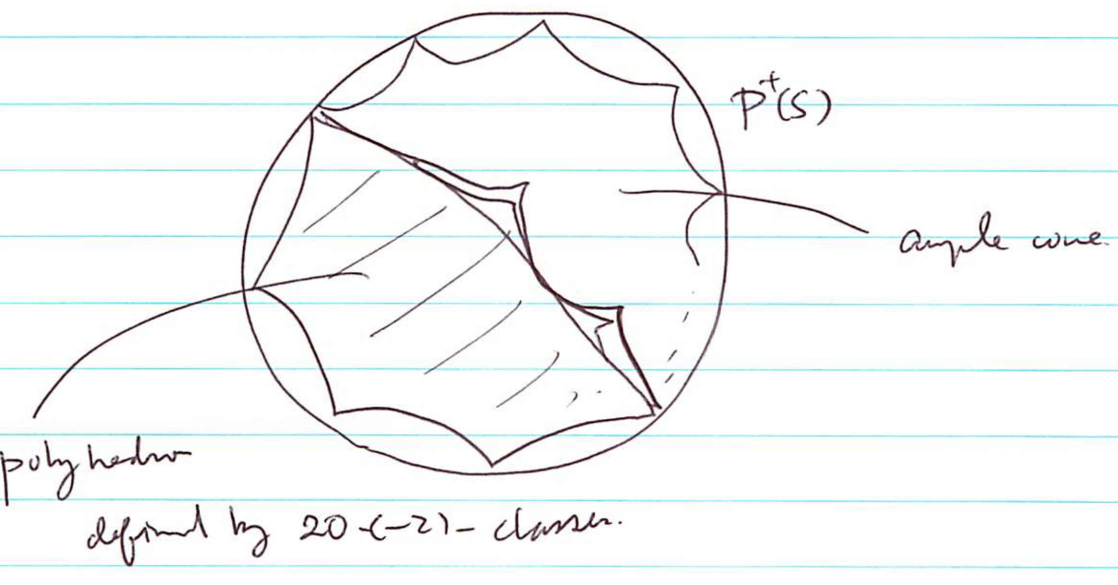
  S is called type VI Enriques surface with

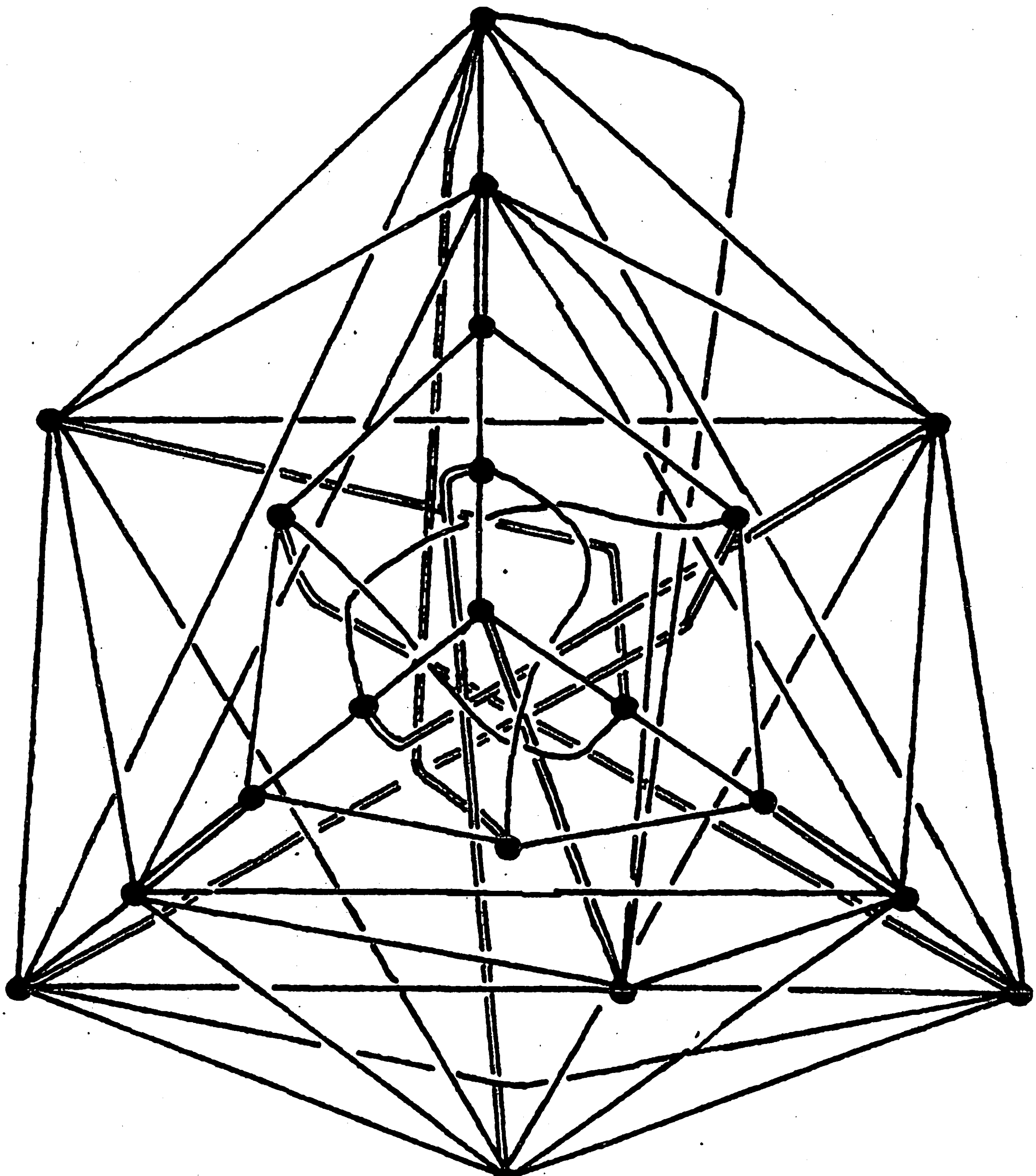
  $\mathrm{Aut}(S) \cong \mathfrak{S}_5$.

- generic case: the projection from a node $P_{ijk} \in H$

  gives us an involution f_{ijk} of X commuting with σ.

  $\mathrm{Aut}(S) \cong \langle f_{P_{ijk}} \rangle$ (Shimada, Ohashi-Mukai)



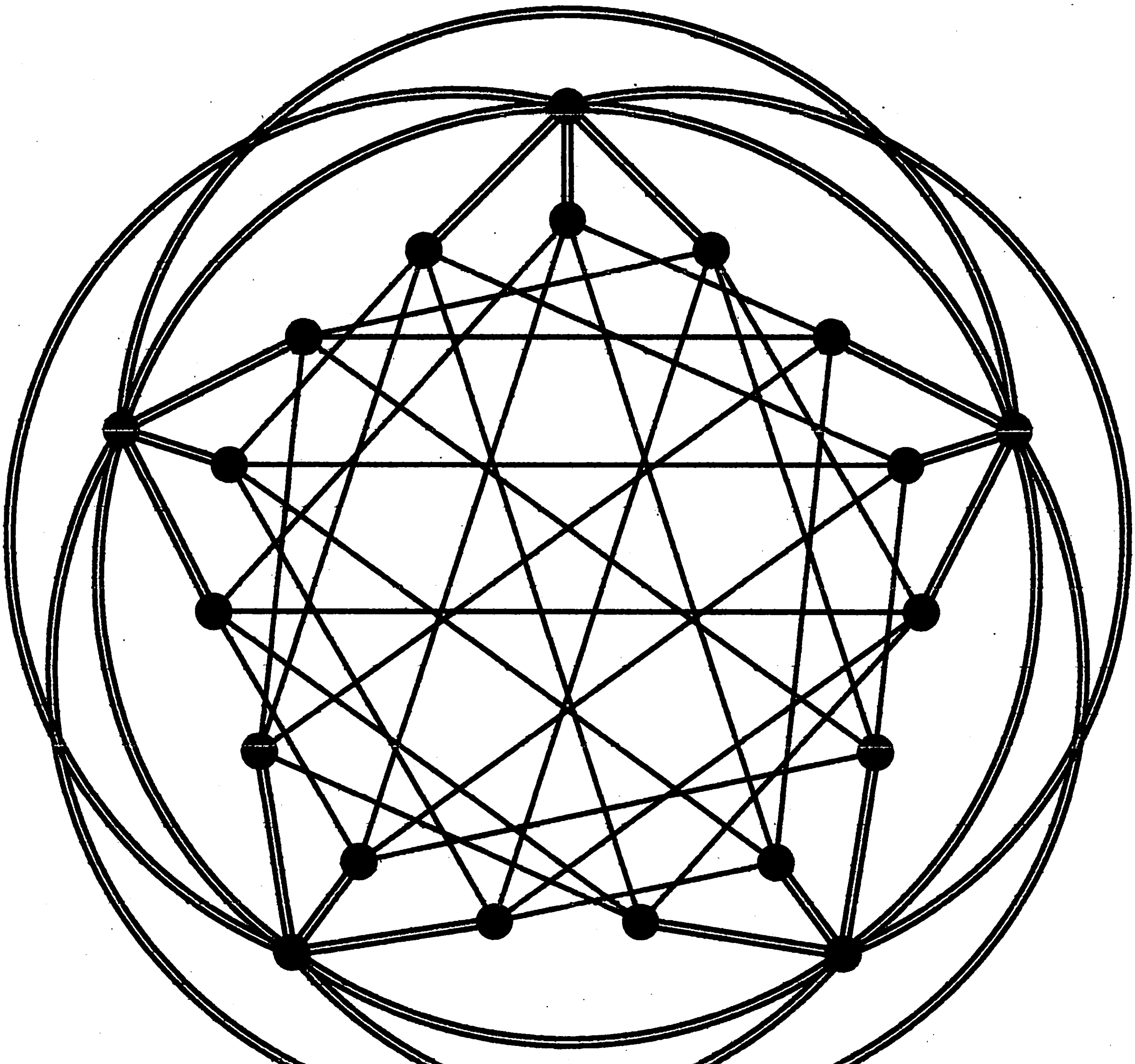


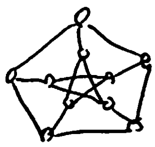
Type VI

Ohashi

$$\sum x_i x_j = \sum \frac{1}{x_i x_j} = 0 \subset \mathbb{P}^4$$

$$(x_i) \longleftrightarrow \left(\frac{1}{x_i}\right)$$





dual.

Type VII $\quad \mathrm{Aut}(S) \cong \mathfrak{S}_5$

given by G. Fano and K — (Nikulin)

§2. Enriques surfaces and Leech lattice.

$\mathrm{II}_{1,25}$: even unimodular lattice of sign $(1,25)$ (unique)

$\mathrm{II}_{1,25} \overset{\text{fix.}}{\cong} U \oplus \Lambda$. $\quad U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$.

Λ = Leech lattice.
= even unimodular negative definite lattice of rank 24 without (-2)-vectors.

$\mathrm{II}_{1,25} \ni x = (m, n, \lambda)$, $\quad m, n \in \mathbb{Z}$, $\lambda \in \Lambda$

$\mathrm{II}_{1,25} \ni \rho = (1, 0, 0)$.

$x^2 = 2mn + \lambda^2$

$\mathrm{II}_{1,25} \ni r = (m, n, \lambda)$ <u>Leech root</u> $\Longleftrightarrow \begin{cases} \cdot\ r^2 = -2 \\ \cdot\ \langle r, \rho \rangle = 1 \end{cases}$

$\rho^{\perp} \not\ni (-2)$-vectors.
$(r = (m, 0, \lambda) \Rightarrow r^2 = \lambda^2 \leqq -4\ )$.

$\Longleftrightarrow r = (m, 1, \lambda)$, $r^2 = -1$. $\Longleftrightarrow r = \left(-1 - \frac{\lambda^2}{2}, 1, \lambda\right)$
$\lambda \in \Lambda$.

$\Delta = \{\text{Leech roots}\} \overset{1:1}{\longleftrightarrow} \Lambda$.

$W := \langle \{ s_\delta \mid \delta \in \mathrm{II}_{1,25}, \delta^2 = -2 \} \rangle \lhd O(\mathrm{II}_{1,25})$.

$\mathcal{C} := \{ x \in P^+(\mathrm{II}_{1,25} \otimes \mathbb{R}) \mid \langle x, r \rangle > 0 \ {}^{\forall} r \in \Delta \}$.

- <u>Thm</u> (Conway) $\mathcal{C}$ is a fundamental domain of W
- R. Borcherds : applied $\mathcal{C}$ to a hyperbolic lattice $L \overset{\text{primitive}}{\hookrightarrow} \mathrm{II}_{1,25}$
- S.K___ : applied to K3 surfaces

- Brandhorst - Shimada : apply to Enriques surface

<u>Thm</u> (2019 B-S).

There are exactly 17 primitive embeddings of $E_{10}(2)$ into $\mathrm{II}_{1,25}$. Under these embeddings (restricting $\mathcal{C}$ to $P^{+}(E_{10}\otimes\mathbb{R})$.)

the following 17 polyhedrons are obtained:

Moreover they showed: except No. 17,

there exists a singular K3 surface (complex K3 surf. with $\rho=20$) whose nef cone is tessellated by the polyhedron given type.

	No.	name	facets	root	Enriques surface	
	1	$12A$	12	D_8	I	
	2	$12B$	12	A_7	II	
	3	$20A$	20	$D_4 + D_5$	V	
	4	$20B$	20	$2D_4$	III	
same	5	$20C$	20	$10A_1 + D_8$	VII in char 2	$12A_1 + D_4$
	6	$20D$	20	$A_3 + A_4$	VII	
	7	$20E$	20	$5A_1 + A_5$	VI	
	8	$20F$	20	$2A_3$	IV	
	9	$40A$	40	$4A_1 + 2A_3$	MIII	$8A_1 + 2A_2$.
	10	$40B$	40	$8A_1 + 2D_4$	MII in char 2	$(8A_1 + D_4) + D_4$
	11	$40C$	40	$6A_1 + A_3$	MII	
	12	$40D$	40	$12A_1 + D_4$	MI in char 2	$12A_1 + D_4$
	13	$40E$	40	$2A_1 + 2A_2$	MI	$2A_1 + 2A_2$.
	14	$96A$	96	$8A_1$	?	
	15	$96B$	96	$16A_1$		
	16	$96C$	96	$4A_1$		
	17	*infty*	infty			

"root" = root sublattice in $E_{10}(2)^{\perp}$ in $\mathrm{II}_{1,25}$.

- In char 2, it has a meaning : rational double points of the canonical coverings.

- In case of Coble surface, # of boundaries.

$$\begin{cases} \mathrm{Pic}(\text{s.s.K3 in char } 2,\ \sigma = 1) \cong D_4^{\perp} \\ \mathrm{Pic}(\quad \text{"} \quad 3,\ \text{"} \quad) \cong (2A_2)^{\perp} \end{cases}$$

§3. Coble surface with finite automorphism group

V : a Coble surface $\iff$ $|-K_V| = \phi$, $|-2K_V| = \{C_1 + \cdots + C_n\}$

$C_i \cong \mathbb{P}^1$, $C_i \cdot C_j = 0$ $(i \neq j)$.

$C_1, \cdots, C_n$: boundary

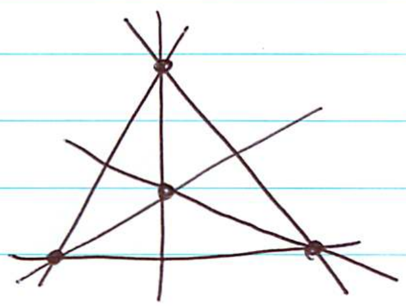
$\iff$ $\exists X \xrightarrow{\pi} V$ X is a K3 surface

Coble surface appear as degeneration of Enriques surface

$/\mathbb{C}$ $D_{10}/\Gamma \supset \mathcal{H}/\Gamma$ $\mathcal{H} = \{$ complements of (-2)-elements $\}$

$(D_{10}/\Gamma) \setminus (\underbrace{\mathcal{H}/\Gamma}_{\text{Coble. surface}})$ = moduli of Enriques.

Ex. (Vinberg–Mukai) char $p \neq 2$.

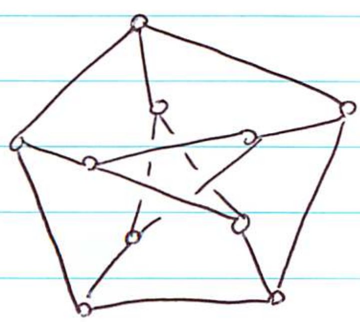


$\mathbb{P}^2$

$\xleftarrow{\text{blow up at 4 points}}$ D_5 = a quintic del Pezzo surface

$\cup$ $\mathrm{Aut}(D_5) \cong \mathfrak{S}_5$

10 lines = 4 exceptional + 6 lines.



dual graph of 10 lines.

$\xleftarrow[\text{15 intersection points}]{\text{blow up at}}$ V $\rho(V) = 1+4+15 = 20$

$|-2K_V| = \{\underbrace{C_1 + \cdots + C_{10}}_{\text{preimage of 10 lines.}}\}$

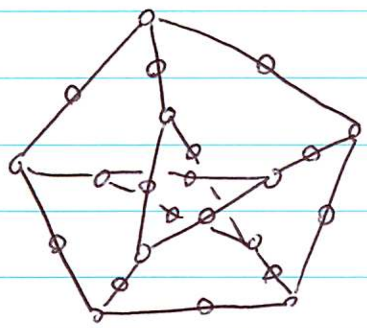
(-4)-vectors.

$V \xleftarrow{2:1} X$ branched along $C_1, \dots, C_{10}$.

$\Rightarrow$ X is a K3 surface with $\rho(X) = 20$. $T_X \cong \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

(Vinberg's Two most alg. K3.) 1983

X contains 25 (-2)-curves. (10 lines + 15 exceptional)

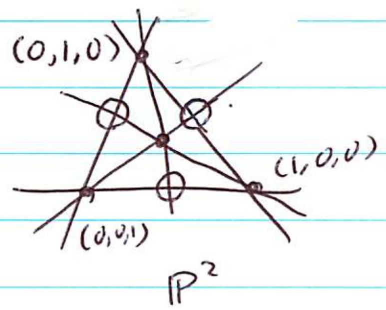


Vinberg found additionally

$\underline{5\ (-4)\text{-vectors.}}$

$\rightsquigarrow$

(-4)-reflections.



Cremona transf.

$$T : (x, y, z) \longrightarrow ((x-y+z)(x-y-z),\ (x+y-z)(x-y-z),\ (z-x-y)(x-y+z))$$

can be lifted to an involution of X.

$\exists\ D_5 \longrightarrow \mathbb{P}^2$ 5 contraction

$\exists$ 5 involutions

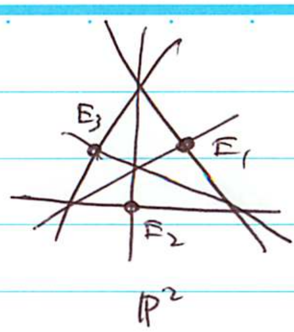
covering transf of $X \to V$.

<u>Th</u> (Vinberg) $\mathrm{Aut}(X) \cong ((\overbrace{\mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}}^{5}) \rtimes \mathfrak{S}_5) \times \mathbb{Z}/2\mathbb{Z}$

<u>Cor</u> $\mathrm{Aut}(V) \cong (\mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}) \rtimes \mathfrak{S}_5$

<u>Remark</u> $\mathcal{C} \,|\, P^+(X)$ is defined by 25 (-2)- and 5 (-4)-vectors.

V has $\begin{cases} 10\ (-4)\text{-curves } C_1, \dots, C_{10} \\ 15\ (-1)\text{-curves} \\ 5\ (-2)\text{-classes } \ell - E_1 - E_2 - E_3. \end{cases}$



Mukai introduced Coble-Mukai lattice

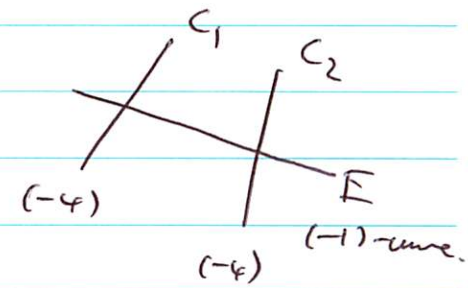
$$\mathrm{Pic}(V) + \tfrac{1}{2}[C_1] + \cdots + \tfrac{1}{2}[C_{10}]$$
$$\cup$$
$$CM(V) := \langle C_1, \dots, C_{10} \rangle^{\perp}$$

K.—— ($/\mathbb{C}$) $\cdot$ $CM(V) \simeq E_{10}$

$\cdot$ conjectured this is true in any p.

$$CM(V) \ni \delta = \tfrac{1}{2}C_1 + \tfrac{1}{2}C_2 + 2E$$

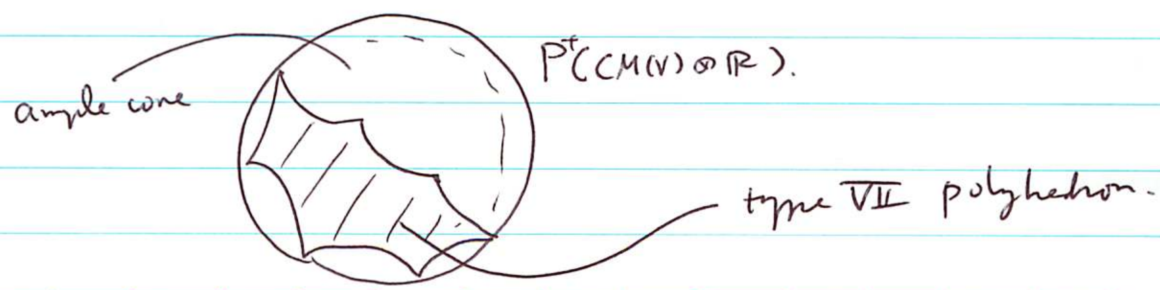
effective. (-2)-class



We obtain:

- 15 effective (-2)-class δ.
- 5 non-effective " $\ell - E_1 - E_2 - E_3$.

forming the dual graph of type VII



$$\mathrm{Aut}(V) \simeq (\mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}) \rtimes \mathfrak{S}_5$$

V exists in char 2

However Cremona transf $T : (x,y,z) \longrightarrow ((x-y+z)(x-y-z), \cdots)$

degenerate.

On the other hand, there exist 5 (-2)-curves:

$$V \supset \begin{array}{ll} K_1 : & x^2+yz=0 \\ K_2 & y^2+xz=0 \\ \vdots & \quad \vdots \\ K_5 & x+y+z=0. \end{array}$$

In char 2, V contain 5 (-2)-curves + 15 δ forming

type VII polyhedron

Thm $\underbrace{}_{p=2}$ $\mathrm{Aut}(V) \cong \mathfrak{S}_5$.

Ex. $\mathfrak{S}_6 \ni (ij)$ 15-transvections (duads) Sylvester

$\ni (ij)(kl)(mn) \quad \{i,j,k,l,m,n\} = \{1,\cdots,6\}$.

$\cup$ 15-involutions (Sylvester synthemes)

$(ijk)(lmn)$ 10-elements of order 3.

$\Rightarrow$ dual graph $\underbrace{}_{\Gamma}$ of 40 vertices. (called the extended Cremona-Richmond polytope)

Brandhorst-Shimada's list (No. 12, 13)

$$\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathfrak{S}_6) \cong \mathfrak{S}_6 \cdot \mathbb{Z}/2\mathbb{Z}$$

$$Q = \mathbb{P}^1(\mathbb{F}_q) \times \mathbb{P}^1(\mathbb{F}_q) \xleftarrow{\text{blow up 10 points } B\cap B'} V$$

Coble with two bdries $\tilde{B}, \tilde{B}'$.

$\cup$

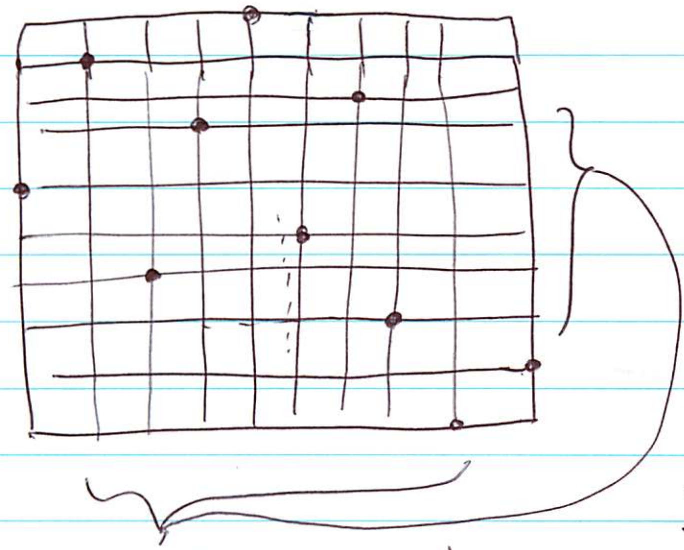
$$\begin{cases} B : u_0 v_0^3 = u_1 v_1^3 \\ B' : u_0^3 v_0 = u_1^3 v_1 \end{cases}$$

$$V \xleftarrow{2:1} X$$

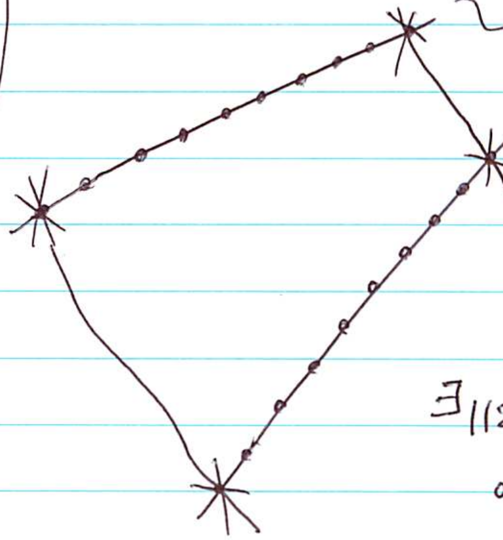
X $K3$ = Fermat quartic surface

$\circlearrowright$ Segre invol.

Q



10 $\mathbb{F}_q$-points on $\mathbb{P}^1(\mathbb{F}_q)$



two skew lines on X

$\cap$

$\mathbb{P}^3$

$\exists$ 112 lines on X.

$$112 = 2 + 10 + 20\times 2 + 30 \times 2$$

bdry, excep., 20 lines, conics.

(ij)

(ij)(kl)(mn) } 30 conics on Q.

(ijk)(lmn) —— 10 exceptional divisors $\frac{1}{2}\tilde{B} + \frac{1}{2}\tilde{B}' + 2E$.

Th. $\mathrm{Aut}(V) \cong \mathrm{Aut}(\mathfrak{S}_6)$

(Mukai-Ohashi, K__)

Type	p	N	k	Aut	R-inv.
I	any	1	12	D_8	$(E_8 \oplus A_1, \{0\})$
I	any	2	12	$(\mathbf{Z}/2\mathbf{Z})^2$	$(E_8 \oplus A_1^{\oplus 2}, \mathbf{Z}/2\mathbf{Z})$
II	any	1	12	$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$	$(D_9, \{0\})$
V	3	2	20	$\mathfrak{S}_3 \times \mathbf{Z}/2\mathbf{Z}$	$(E_7 \oplus A_2 \oplus A_1^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^2)$
VI	5	1	20	$\mathfrak{S}_5$	$(E_6 \oplus A_4, \{0\})$
VI	3	5	20	$\mathfrak{S}_5$	$(E_6 \oplus D_5, \mathbf{Z}/2\mathbf{Z})$
VII	5	1	20	$\mathfrak{S}_5$	$(A_9 \oplus A_1, \mathbf{Z}/2\mathbf{Z})$
MI	3	2	40	$\mathrm{Aut}(\mathfrak{S}_6)$	$(A_5^{\oplus 2} \oplus A_1^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^3)$
MII	3	8	40	$(\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes \mathbf{Z}/2\mathbf{Z}$	$(D_8 \oplus A_2^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^2)$

TABLE 1. Coble surfaces with finite automorphism group ($p \neq 2$)

Type	p	k	Aut	R-inv.	Moduli
I	any	12	D_8	$(E_8 \oplus A_1, \{0\})$	$\mathbb{A}^1 \setminus \{0, -2^{10}\}$
II	any	12	$\mathfrak{S}_4$	$(D_9, \{0\})$	$\mathbb{A}^1 \setminus \{0, -2^8\}$
III	any	20	$(\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_8$	$(D_8 \oplus A_1^{\oplus 2}, (\mathbf{Z}/2\mathbf{Z})^2)$	unique
IV	any	20	$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$	$(D_5^{\oplus 2}, \mathbf{Z}/2\mathbf{Z})$	unique
V	$\neq 3$	20	$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$	$(E_7 \oplus A_2 \oplus A_1, \mathbf{Z}/2\mathbf{Z})$	unique
VI	$\neq 3, 5$	20	$\mathfrak{S}_5$	$(E_6 \oplus A_4, \{0\})$	unique
VII	$\neq 5$	20	$\mathfrak{S}_5$	$(A_9 \oplus A_1, \mathbf{Z}/2\mathbf{Z})$	unique

TABLE 2. Enriques surfaces with finite automorphism group ($p \neq 2$)

Congratulation on

70 years old!