Automorphic Forms and Quantum Field Theory

Werner Nahm
School of Theoretical Physics, DIAS, Dublin 4, Ireland
wnahm@stp.dias.ie

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Abstract

The partition functions of euclidean quantum field theory can be described as maps from the space of compact manifolds with Riemannian metric that have few derivatives. Their fields are just those derivatives. This generalizes Einstein's characterization of the energy-momentum tensor as the affine derivative with respect to the metric. For conformal field theories in two dimensions the dependence on Weyl transformations can be factored out, at the price of introducing an automorphy factor for the action of the mapping class group. In a simple case, this leads to a generalization of the Rogers-Ramanujan functions to arbitrary genus.

1 Euclidean Quantum Fields: the theory of very smooth partition functions

Some 40 years ago, Don Zagier asked me three questions: What is quantum field theory? What is a field? Why are physicists interested in automorphic forms, in particular in modular forms? I hope that my attempts to answer these questions are becoming comprehensible for mathematicians. First, fix a dimension d.

Let $\mathcal{R}$ be the set of compact d-dimensional manifolds M with Riemannian metric g (up to isomorphisms). Partition functions are maps

$$Z : \mathcal{R} \to \mathbb{R}_+.$$

Any (specimen of) euclidean quantum field theory is determined by its partition function.

One example is

$$Z = det(\Delta_g + m^2)^{-1/2},$$

where Δ_g is the Laplacian for the metric g and m is a real number. This defines what is called the quantum field theory of a free scalar of mass m.

Given a partition function Z, what are the corresponding fields? On $\mathcal{R}$ there is a natural notion of derivative, modelled on the derivatives of functions on affine spaces $\mathcal{A}$. Partition functions must be smooth in the sense of admitting arbitrary derivatives. For given Z, two derivatives are considered equivalent when they have the same action on Z. The equivalence classes are called the fields of Z.

We shall consider axioms for quantum field theory. For $d = 1$ they have no physical meaning, but provide a useful perspective. In this case $\mathcal{R}$ consists of circles characterized by their circumference x and disjoined unions of such circles. The axioms imply that $Z(x)$ satisfies a linear differential equation with constant coefficients. In other words, Z has finite dimensional spaces of derivatives modulo equivalence. In a sense, the partition functions for $d = 1$ are the simplest smooth elementary functions. Partition functions of euclidean quantum field theories are characterized by a similar property. We will say that they are very smooth.

Though very smooth functions are the simplest smooth functions on $\mathcal{R}$, their investigation is still in its infancy. For $d = 2$, however, in many cases Z is known when the genus of M is restricted to 0 or 1. For genus 1 one may suppose that M is the torus $\mathbb{C}/\langle 1, \tau\rangle$ with volume V and metric

$$g = \exp(\chi) dz d\bar{z}.$$

Then in one case, called the (2,5) minimal model

$$Z(M) = \exp\left(-\frac{11}{120\pi} \int \chi K \right) (|f(\tau)|^2 + |g(\tau)|^2),$$

where f, g are the Rogers-Ramanujan modular functions and

$$K = -\partial\bar{\partial}\chi$$

is the Gauss curvature. Modular forms come into play, since

$$det'(\Delta_g) = V \exp\left(-\frac{1}{24\pi} \int \chi K \right) Im(\tau) |\eta(\tau)|^4.$$

1.1 Derivatives

On an affine space $\mathcal{A}$ a natural operation is the translation by a vector v. Moreover, vectors can be scaled by $\epsilon \in \mathbb{R}_+$. Regard each vector as a basis element $[v]$ of a very big vector space and consider finite formal sums

$$A = \sum_i a_i[v_i],$$

where the a_i are real numbers. For $A \neq 0$ there is a unique maximal number $k \in \mathbb{N}$ so that

$$D_k(A)f(x) = \lim_{\epsilon \to 0} \epsilon^{-k} \sum_i a_i f(x + \epsilon v_i)$$

exists for any smooth $f : \mathcal{A} \to \mathbb{R}$. We write $D(A) = D_k(A)$ for this derivative of order k. When $D(A^1) = D(A^2)$ then $D(A^1 - A^2)$ is a derivative of higher order.

We want to define derivatives for partition functions based on local changes of elements of $\mathcal{R}$. A natural operation is cutting and glueing. Let B be a manifold with Riemannian metric that has the unit sphere in R^d as boundary. Such manifolds can be scaled by $\epsilon \in \mathbb{R}_+$ so that the boundary becomes the sphere with radius ϵ. Given a chart of M and a frame at a point x in this chart one can cut out a small sphere of radius ϵ in M and glue in the rescaled B. The result will be called $B\epsilon M(x)$. Consider formal sums $\mathcal{B} = \sum_i a_i[B_i]$. Local derivatives $D(\mathcal{B})$ of order $k \in \mathbb{R}$ are defined as before, as limits

$$\lim_{\epsilon \to 0} \epsilon^{-k} \sum_i a_i Z(B_i \epsilon M(x)).$$

One precision concerning $\mathcal{R}$: One needs $B\epsilon M(x) \in \mathcal{R}$, thus we admit metrics which are continuous and piecewise smooth, where the pieces are submanifolds of M with smooth boundaries. Nevertheless, values of derivatives of Z will only be considered at smooth points of $\mathcal{R}$.

Multilocal derivatives are defined analogously, with metric manifolds B having m boundary components, all given by unit spheres. After rescaling they can be glued in around m distinct points $x_1, \ldots, x_m$ of M.

If B^1, B^2 have m^1, m^2 boundary components respectively, then $B^1 \sqcup B^2$ has $m^1 + m^2$ boundary components. One has

$$(B^1 \sqcup B^2)\epsilon M = B^1\epsilon(B^2\epsilon M),$$

with an obvious choice of the insertion points. The glueing operation is commutative, so that fields at distinct points should be given by commuting derivative operators.

1.2 Axioms

Partition functions have to satisfy four axioms. Axiom 1 states that they are multiplicative with respect to disjoint union:

$$Z(M_1 \sqcup M_2) = Z(M_1)Z(M_2).$$

As in this case, we often suppress one or both arguments of $Z(M, g)$.

Secondly, partition functions must be smooth. This is expressed by the axioms 2a and 2b, though one hopes that they might be combined when the theory is understood better.

Axiom 2a uses the fact that the space of metrics on a manifold is locally affine. It states that Z is smooth with respect to the corresponding derivatives.

Axiom 2b states that for every non-zero $\mathcal{B}$ there should be a non-zero derivative $D(\mathcal{B})$ acting on Z.

By abuse of notation, the corresponding field (equivalence class) will be denoted by the same expression. The order of $D(\mathcal{B})$ is called the scaling dimension of the field.

Axiom 3 states that Z is very smooth in the following sense. There exists an integer m_0 such that manifolds B with less than m_0 boundary components and their disjoint unions are sufficient to obtain all multi-local fields.

1.3 The case d=1

For $d = 1$ axiom 1 implies that Z is determined by its values on circles of circumference x. By axiom 2 it must be a smooth function of x. To use Axiom 3 let B be a circle with two intervals cut out. One obtains

$$Z(x_1 + x_2) = \sum_{i,j} a_{ij} \frac{d^i Z(x_1)}{dx_1^i} \frac{d^j Z(x_2)}{dx_2^j},$$

where the sum is finite. This implies that Z satisfies a linear ODE with constant coefficients.

The case $d = 1$ is useful for illustration, but of no physical importance. From here on we consider $d > 1$.

1.4 Notation

The definition of a field $\phi = D(\mathcal{B})$ uses a chart U with coordinate x. By abuse of notation we do not distinguish between points in U and their coordinates. When we write $\phi(x)$ we refer to the whole family of derivatives for points in the chart.

Let $\phi_1, \ldots, \phi_n$ be fields. Physicists use the notation

$$\langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle = \phi_1(x_1) \ldots \phi_n(x_n) Z,$$

and $\langle 1 \rangle = Z$ in the case $n = 0$. These functions are called n-point functions.

By axiom 2a the n-point functions are smooth as long as the x_i are distinct.

1.5 The energy-momentum tensor

According to axiom 2a one can define the standard functional derivative $\delta Z/\delta g_{\mu\nu}(x)$ of Z with respect to the metric

$$g = g_{\mu\nu} dx^\mu dx^\nu.$$

Following Einstein, it is denoted by

$$-\frac{1}{2} T^{\mu\nu}(x) = \frac{\delta}{\delta g_{\mu\nu}}(x).$$

Thus

$$Z(g + \epsilon h) = Z(g) - \frac{\epsilon}{2} \int h_{\mu\nu}(x)\, T^{\mu\nu}(x) Z\ dV + O(\epsilon^2),$$

with the volume measure dV given by the metric g.

Equivalently, let B_1, B_2 be unit balls with metrics g^1, g^2 and corresponding volume elements dV^1, dV^2. Then $D(B_1 - B_2)$ is a field of dimension d with

$$D(B_1 - B_2)(x) = -\frac{1}{2} T^{\mu\nu}(x) \int (g^1_{\mu\nu} dV^1 - g^2_{\mu\nu} dV^2).$$

The summation over μ, ν is implied (Einstein convention).

By definition, the field $T^{\mu\nu}$ is called the energy-momentum tensor. Under coordinate transformation it transforms dually to the metric.

By definition, the elements of $\mathcal{R}$ are defined up to isomorphism. Thus a change of the metric by reparametrization must not change Z. For infinitesimal reparametrizations this is expressed by

$$D_\mu T^{\mu\nu} = 0,$$

where D_μ is the covariant derivative on M with respect to the Levi-Civita connection for g.

A euclidean quantum field theory is called conformal, if Z transforms canonically under Weyl transformations (rescalings of the metric g by a factor $\exp \chi(x)$). This means that the trace $g_{\mu\nu} T_{\mu\nu}$ of the energy-momentum tensor must be canonical. For $d = 2$

$$g_{\mu\nu} T^{\mu\nu} = \frac{c}{24\pi} R,$$

where R is the curvature of the metric and the number c is called the central charge. From now on we only will consider theories of this kind.

Consider two metrics related by a Weyl transformation,

$$g^2 = \exp(\chi) g^1.$$

One obtains

$$\log(Z(g^2)/Z(g^1)) = \frac{c}{48\pi} \int \chi(K^1 + K^2),$$

where K^1, K^2 are the Gauss curvature forms of g^1, g^2. When the genus of M is zero, then any two metrics are related by an isomorphism and a Weyl transformation, so Z is fixed up to a factor, which can be chosen according to convenience.

In complex coordinates z the metric takes the form $g = 2\exp(\chi)dzd\bar{z}$. When one defines

$$\frac{T(z)}{2\pi} = T_{zz} + \frac{c}{24\pi}\left(\frac{\partial^2\chi}{\partial z^2} - \frac{1}{2}\left(\frac{\partial\chi}{\partial z}\right)^2\right)$$

in a conformal theory with central charge c, then $T(z)$ commutes with Weyl transformations and the invariance of Z with respect to reparametrizations implies that $T(\,z)$ is a holomorphic function of z.

Consider a holomorphic vector field $v(z)\partial_z$ in a tubular neighborhood of a closed curve C on M. One can change the complex structure of M by cutting along C, changing the complex coordinate on the left side according to v and glueing back. Up to possible curvature terms the corresponding change of Z is given by the action of

$$\oint_C vT\frac{dz}{2\pi i} + \text{ complex conjugate.}$$

For constant v this operator describes the translation and reglueing of the interior of C. In particular, the all insertion points of fields within C will be translated. Thus the n-point functions of any fields are real analytic.

When one considers non-contractable curves C, it follows that derivatives of Z with respect to the complex moduli of M are given by the 1-point function of T, Derivatives of the 1-point function are given by the 2-point function etc. In the minimal models a finite set of n-point functions determines all of them, so that one obtains a system of ODEs for Z with respect to the complex moduli.

2 Z for hyperelliptic M in a minimal model

When B has genus zero and one boundary circle, then $B\epsilon M(x)$ has the same genus as M. Thus for genus 0 all n-point functions of the field T can be calculated. For $c = -22/5$ one finds

$$T(z_1)T(z_2) = \frac{c}{2}(z_1 - z_2)^{-4} + (T(z_1) + T(z_2))(z_1 - z_2)^{-2} - \frac{1}{5}T''(z_1) + O(z_1 - z_2).$$

The singular terms are generic, but only for $c = -22/5$ can the order zero term be written as a derivative of T. The form is invariant under holomorphic coordinate changes. The (2,5) minimal model is defined by the validity of this relation for M of any genus.

Let M be given by a hyperelliptic curve $y^2 = p(x)$ with a monic polynomial p of odd order N. As metric use $g = dxd\bar{x}$ outside of small circles around the ramification points and complete it by the flat metric inside these circles.

Let $T^e(x)$ be the even part of T with respect to the sign change of y. Then

$$\theta = pT^e - \frac{c}{32}\frac{p'^2}{p}$$

is regular at finite x and satisfies

$$\theta(x) = \frac{c}{32}(1 - N^2)x^{N-2} + O(x^{N-3})$$

at ∞. The singularities of the n-point functions at the partial diagonals $x_i = x_j$ are known, so that these functions are determined up to polynomials of order $N - 3$ in each variable. In the (2,5) minimal model the restrictions of these polynomials to the partial diagonals are known by recursion. Thus all the polynomials are determined by F_N coefficients, where the F_N are the Fibonacci numbers $F_3 = 2$, $F_5 = 5$ etc. With respect to each zero of p, Z satisfies an ODE of order F_N (joint work with Marianne Leitner).

Let X_i, $i = 1, \ldots, N$ be the zeros of p. For $N = 3$ one obtains

$$D_i Z = \frac{2}{p'(x)} \langle \theta(x) \rangle |_{X=X_i}$$

where

$$D_i = \frac{\partial}{\partial X_i} - \frac{c}{8} \sum_{j \neq i} \frac{1}{X_i - X_j}$$

and

$$D_i \langle \theta(x_i) \rangle = \left(\frac{7c}{320} \frac{(p'')^2}{p'} (X_i) - \frac{27c}{80} \right) Z + \frac{9}{10} \langle \theta(X_i) \rangle .$$

When one fixes two zeros at 0,1, this is a (2,1) hyperelliptic ODE with parameters $\frac{3}{10}, -\frac{1}{10}, \frac{3}{5}$. This is a prominent case in Schwarz' study of hyperelliptic ODEs with algebraic solutions. When one changes to the flat torus metric, the solutions become the Rogers-Ramanujan modular functions. There are two of them since $F_3 = 2$.

For higher N the equations become more complicated, since the Fibonacci numbers increase exponentially. The coefficients are always differential polynomials in p divided by p', so the singularities are regular. The Frobenius indices remain the same. When three ramification points approach each other, M splits into into a manifold of lower genus and a torus $\mathbf{T}$ and the F_N-dimensional solution space decomposes into F_{N-1} and F_{N-2}-dimensional subspaces. The F_{N-2}-dimensional subspace just corresponds to the splitting $M = M' \sqcup \mathbf{T}$.The F_{N-1}-dimensional subspace describes the insertion of a field $\Phi = D(\mathbf{T}')$, where the prime denotes the removal of a disk from $\mathbf{T}$. Changing the shape of $\mathbf{T}'$ yields the same field up to a multiplicative constant.

When only two ramification points approach each other one obtains a cylinder inserted at two points x, y and $x, -y$. Up to a constant the result is the same as for $\mathbf{T}' \sqcup \mathbf{T}'$. Indeed the (2,5) minimal model satisfies axiom 3 with $m_0 = 2$. The scaling dimension of Φ is $-2/5$, given by the corresponding Frobenius index.

For derivatives on affine spaces there no negative orders, indeed we have $D([v]) = 1$ for any vector v. In unitary quantum field theories one also has $D([B]) = 1$ for any B. In general this is only true when B has genus zero.

Negative scaling dimensions indicate instabilities when one tries to go from euclidean space to Minkowski space by analytic continuation. But any Minkowskian theory should yield a euclidean one, so our axioms constitute a reasonable starting point for mathematicians who want to work in quantum field theory.