SCHLESINGER ISOMONODROMIC DEFORMATION

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Abstract. We consider the isomonodromic criterion of the family of Fuchsian systems and Schlesinger isomonodromic deformation. We get Schlesinger system of equations and understand how to get the sixth Painlevé equation from it. The contents of this talk refers to the book [Bol09].

Contents

1. Monodromy and isomonodromic criteria

1.1. **Definitions.** Let us consider Fuhcsian connection

$$\nabla \frac{d}{dz} y = \frac{dy}{dz} - B(z)\, y, \quad y \in \mathbb{C}^p,$$

and the equation corresponding to it

$$\frac{dy}{dz} = B(z)\, y. \tag{1}$$

Let us consider a point $z_0 \in U(0) \setminus \{z_0\} = \mathring{U}(0)$ and some matrix of fundamental solutions $Y(z)$ of the system (1) in a neighborhood of z_0.

For each loop $\gamma \in \mathring{U}(0)$, which starts at z_0, the matrix $Y(z)$ extends analytically along γ. The result of such continuation is a matrix $\tilde{Y}(z)$. Since two fundamental matrices are dependent in a neighborhood of a singular point, we have

$$\tilde{Y}(z) = Y(z)\, G_\gamma, \quad G_\gamma \in GL_p(\mathbb{C}).$$

Therefore, there is a homomorphism from fundamental group of loops to non-degenerative matrices

$$\chi_0 : \pi_1\left(\mathring{U}(0), z_0\right) \to GL_p(\mathbb{C}). \tag{2}$$

The map (2) is a local monodromy representation at the point $z = 0$.

Example 1.

$$\frac{dy}{dz} = \begin{pmatrix} \frac{1}{z} & 1 \\ 0 & 0 \end{pmatrix} y, \qquad Y(z) = \begin{pmatrix} z & z\ln z \\ 0 & 1 \end{pmatrix}.$$

Analytical continuation along the loop γ around the point $z = 0$ is

$$\tilde{Y}(z) = \begin{pmatrix} z & z\ln z + 2\pi i z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & z\ln z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}.$$

Therefore, the monodromy matrix is

$$G_\gamma = \chi_0(\gamma) = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}.$$

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Now we will interested in deformations which preserve monodromy matrices, i.e. isomonodromic deformations. The isomonodromic problem is a problem of an embedding of the system (1) into a family of systems which are holomorphic in a parameter $b=(b_1,\dots,b_k)\in\mathbb{C}^k$

$$\frac{dy}{dz}=B\left(z,b\right)y,\quad y\in\mathbb{C}^p, \tag{3}$$

and coincides with (1) when $b=b^0$.

Definition 1. The system (3) is a *deformation* of the system (1).

Remark. Intuitively, the system (3) is an isomonodromic deformation if for each fixed parameter b the system (3) has the monodromy matrix which coincides with a monodromy matrix of system (1), i.e. when $b=b^0$. But if parameter b is different, then a family of singular points of the system (3) is different too. On the other hand, the monodromy

$$\chi:\pi_1\left(\bar{\mathbb{C}}\setminus\{a_1,\dots,a_n\}\right)\to GL_p\left(\mathbb{C}\right) \tag{4}$$

has different domains $\bar{\mathbb{C}}\setminus\{a_1,\dots,a_n\}$ for different values of the parameter b. Hence, we should precise when maps (4) coincide.

Remark. The most interesting case is when the parameter b is a set of singular points of the system (3), i.e. $a=(a_1,\dots,a_n)$.

So, let us consider Fuchsian system of p linear differential equations with singular points $a_1,\dots,a_n\in\mathbb{C}$ on the Riemann sphere

$$\frac{dy}{dz}=\left(\sum_{i=1}^{n}\frac{B_i^0}{z-a_i}\right)y,\quad y\in\mathbb{C}^p,\quad \sum_{i=1}^{n}B_i^0=0,$$

and the family of systems (3)

$$\frac{dy}{dz}=\left(\sum_{i=1}^{n}\frac{B_i\left(a\right)}{z-a_i}\right)y,\quad y\in\mathbb{C}^p,\quad B_i\left(a^0\right)=B_i^0,\quad \sum_{i=1}^{n}B_i=0, \tag{5}$$

which depends holomorphic in the parameter $a=(a_1,a_n)\in D\left(a^0\right)$, where $D\left(a^0\right)$ is a local ball in the space $\mathbb{C}^n\setminus\bigcup\limits_{i\neq j}\{a_i=a_j\}$. We do not include diagonals $\{a_i=a_j\}$ since we consider singular points which change their position and do not coincide with each other.

The family (5) is on the space

$$T=\left(\bar{\mathbb{C}}\times D\left(a^0\right)\right)\setminus\bigcup_{i=1}^{n}\{z-a_i=0\},\quad z\in\bar{\mathbb{C}}.$$

We will look at the set of loops $g_1^0,\dots,g_n^0\in\bar{\mathbb{C}}\setminus\left\{a_1^0,\dots,a_n^0\right\}$, which start at z_0 and generate the group $\pi_1\left(\bar{\mathbb{C}}\setminus\left\{a_1^0,\dots,a_n^0\right\},z_0\right)$.

Let $g_1^a,\dots,g_n^a$ be loops on the space $\bar{\mathbb{C}}\setminus\{a_1,\dots,a_n\}$, which begin from z_0 and generate the fundamental group $\pi_1\left(\bar{\mathbb{C}}\setminus\{a_1,\dots,a_n\},z_0\right)$. Let each of such loops $t_a g_i^a t_a^{-1}$ on the space T be homotopic to a corresponding loop g_i^0, where t_a is a path from $\left(z_o,a^0\right)$ to (z_0,a) on T.

If the parameter a changes a bit, then such homotopies are possible due to the fact that the space T is represented as the set of n-punctured balls, which are contractible on each of their stalks. Therefore, we have a canonical isomorphism between fundamental groups

$$\pi_1\left(\bar{\mathbb{C}}\setminus\{a_1,\dots,a_n\},z_0\right)\simeq\pi_1\left(\bar{\mathbb{C}}\setminus\left\{a_1^0,\dots,a_n^0\right\},z_0\right).$$

Definition 2. The family (5) is called *isomonodromic* if for each fixed $a\in D\left(a^0\right)$ the corresponding system (5) has the same monodromy matrix as a monodromy matrix with parameter $a=a^0$ (up to the homotopic classes of g_i^a and a_i^0 respectively).

Therefore, for each parameter a there is a fundamental matrix $Y\left(z,a\right)$ which corresponding to the system (5) and which has similar monodromy matrix as the monodromy matrix with parameter $a=a^0$ (up to a loop g_i^a) for each parameters $a\in D\left(a^0\right)$. Hence, $Y\left(z,a\right)$ is an *isomonodromic* family of matrices.

Example 2. Let us look at the family

$$\frac{dy}{dz}=\left(\sum_{i=1}^{n}\frac{b_i\left(a\right)}{z-a_i}\right)y,\quad \sum_{i=1}^{n}b_i\left(a\right)=0,\quad y\in\mathbb{C}^1.$$

The solution is

$$y\left(z,a\right)=c\left(a\right)\prod_{k=1}^{n}\left(z-a_k\right)^{b_k(a)},$$

where the factor corresponding to a loop g_k^a equals $\exp\left(2\pi i b_k\left(a\right)\right)$. This family is isomonodromic if $b_k\left(a\right)\equiv$ const.

1.2. **Isomonodromic criteria.**

Theorem 1. *The family (5) of Fuchsian systems is isomonodromic if and only if on the space T there is a matrix holomorphic differential 1 - form Ω such that*

(1) $\Omega|_a=\omega=\sum_{i=1}^{n}\frac{B_i\left(a\right)}{z-a_i}dz$ *for each* $a\in D\left(a^0\right)$,
(2) $d\Omega=\Omega\wedge\Omega$.

Proof. $(\Rightarrow)$
Let us consider analytic isomonodromic fundamental matrix $Y\left(z,a\right)$ of the family (5). Then matrix differential $1-$ form is

$$\Omega=dY\left(z,a\right)Y^{-1}\left(z,a\right).$$

(1) For each loop $g\in\pi_1\left(T,\left(z_0,a\right)\right)$ an analytic continuation $g^*\Omega$ of the form Ω along a loop g is

$$\begin{aligned}g^*\Omega&=d\left(g^*Y\right)\left(g^*Y\right)^{-1}\\&=\left(dYG_g\right)\left(G_g^{-1}Y^{-1}\right)\\&=\left(dY\right)Y^{-1}\\&=\Omega.\end{aligned}$$

Since for each fixed point $a\in D\left(a^0\right)$ the matrix $Y\left(z,a\right)$ is a fundamental matrix of the corresponding system of the family (5), we have

$$\frac{dY\left(z,a\right)}{dz}=\left(\sum_{i=1}^{n}\frac{B_i\left(a\right)}{z-a_i}\right)Y\left(z,a\right).$$

(2)

$$\begin{aligned}d\Omega&=\left(d^2Y\right)Y^{-1}-dY\wedge d\left(Y^{-1}\right)\\&=-dY\wedge d\left(Y^{-1}\right)\\&=dY\wedge Y^{-1}\left(dY\right)Y^{-1}\\&=\left(dY\right)Y^{-1}\wedge\left(dY\right)Y^{-1}\\&=\Omega\wedge\Omega.\end{aligned}$$

$(\Leftarrow)$
Let Ω be a matrix holomorphic differential $1-$ form on the space T, which satisfies theorem conditions. Let us consider a complex manifold

$$M=T\times GL_p\left(\mathbb{C}\right)$$

with coordinates $\left(\left(z,a\right),Y\right)$, and p^2 differential equations on M in the following form

$$dY-\Omega Y=0. \tag{6}$$

According to Frobenius theorem, for each point $\left(\left(z_0,a^0\right),Y^0\right)\in M$ the equation (6) has a local solution $Y=Y\left(z,a\right)$ with initial condition $Y\left(z_0,a^0\right)=Y^0$ if matrix holomrphic differential $1-$ form $\Theta=\left(\theta_{ij}\right)=$

$dY - \Omega Y$ satisfies the total integrability condition, i.e. for each elements θ_{ij} its differential can be represented i the form $d\theta_{ij} = \sum_{k,l} \alpha_{kl} \wedge \theta_{kl}$, where α_{kl} are certain holomorphic on M differential $1-$ forms.

$$\begin{aligned} d\Theta &= d^2 Y - d(\Omega Y) \\ &= (-d\Omega) Y + \Omega \wedge dY \\ &= (-d\Omega) Y + \Omega \wedge (\Theta + \Omega Y) \\ &= (-d\Omega + \Omega \wedge \Omega) Y + \Omega \wedge \Theta \\ &= \Omega \wedge \Theta. \end{aligned}$$

Since Ω satisfies the first theorem condition, the fundamental matrix $Y(z,a)$ of the equation (6) is a fundamental matrix of the family (3) at each fixed point $a \in D(a^0)$. Moreover, the matrix $Y(z,a)$ is isomonodromic since for each point $a \in D(a^0)$ it has the same monodromy matrix with respect to loops $g_i^a \in \pi_1(\bar{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0)$. These monodromy matrices coincide with monodromy matrices which are corresponding to the analytic continuation of $Y(z,a)$ as a function on the space T along the loops, which loop hyperplanes $\{z - a_i = 0\}$. Therefore, the family (3) is the isomonodromic family. □

Remark. The Isomonodromic family (3) is completely defined by the $1-$ differential form Ω, but ambiguous. We can define another $1-$ form $\tilde{\Omega} = \Omega + df(a)\mathbb{1}$, where $f(a)$ is a holomorphic in $D(a^0)$ function. Hencce,

$$d\tilde{\Omega} = d\Omega,$$

and

$$\begin{aligned} \tilde{\Omega} \wedge \tilde{\Omega} &= \Omega \wedge \Omega + df \wedge \Omega + \Omega \wedge df + (df \wedge df)\mathbb{1} \\ &= \Omega \wedge \Omega. \end{aligned}$$

2. Schlesinger isomonodromic deformation

We are interested in Fuchsian system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i^0}{z - a_i} \right) y, \tag{7}$$

and its deformation of the following form

$$\omega_s = \sum_{i=1}^{n} \frac{B_i(a)}{z - a_i} d(z - a_i). \tag{8}$$

Definition 3. The differential $1-$ form (8) is called *Schlesinger deformation.*

First of all, we should show that such isomonodromic deformation (8) of the system (7) exists, i.e. there is the following problem

$$d\omega_s = \omega_s \wedge \omega_s, \quad B_i(a_i^0) = B_i^0.$$

Remark. Obviously, the first criteria of the theorem 1 holds.

Theorem 2. $d\omega_s = \omega_s \wedge \omega_s \Leftrightarrow$

$$dB_i(a) = -\sum_{j \neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j} d(a_i - a_j), \quad i = \overline{1, n}.$$

Proof. $(\Rightarrow)$

Since $d\left(\frac{d(z-a_i)}{z-a_i}\right) = -\frac{d(z-a_i)}{(z-a_i)^2} \wedge d(z-a_i) = 0$, then

$$\begin{aligned} d\omega_s = d\left(\sum_{i=1}^{n} \frac{B_i(a)}{z-a_i} d(z-a_i) \right) &= \sum_{i=1}^{n} dB_i \wedge \frac{d(z-a_i)}{z-a_i} \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} d(a_j - z + z) \right) \wedge \frac{d(z-a_i)}{z-a_i} \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} dz \right) \wedge \frac{d(z-a_i)}{z-a_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} \frac{d(z-a_i) \wedge d(z-a_j)}{z-a_i}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\omega_s \wedge \omega_s &= \left(\sum_{i=1}^{n} \frac{B_i(a)}{z-a_i} d(z-a_i)\right) \wedge \left(\sum_{j=1}^{n} \frac{B_j(a)}{z-a_j} d(z-a_j)\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} B_i B_j \frac{d(z-a_i)\wedge d(z-a_j)}{(z-a_i)(z-a_j)} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} B_i B_j \frac{1}{a_i-a_j}\left(\frac{1}{z-a_i}-\frac{1}{z-a_j}\right) d(z-a_i)\wedge d(z-a_j).\end{aligned}$$

Since $dz \wedge d(z-a_1) \wedge \cdots \wedge d(z-a_n) = (-1)^n dz \wedge da_i \wedge \cdots \wedge da_n \neq 0$, $2-$ forms $dz \wedge d(z-a_i)$, $d(z-a_i) \wedge d(z-a_j)$, $i<j$, are linearly independent over the field of the meromorphic functions in the space $\bar{\mathbb{C}} \times D(a^0)$.

By comparing the coefficients at the corresponding $2-$ forms $dz \wedge d(z-a_i)$, $d(z-a_i) \wedge d(z-a_j)$, we have

$$\sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} = 0,$$

and due to the antysymmetry of the wedge product,

$$\frac{\partial B_i}{\partial a_j}\frac{1}{z-a_i} - \frac{\partial B_j}{\partial a_i}\frac{1}{z-a_j} = \frac{[B_i,B_j]}{a_i-a_j}\left(\frac{1}{z-a_i}-\frac{1}{z-a_j}\right), \quad i \neq j, \quad i,j=\overline{1,n},$$

$$\left(\frac{\partial B_i}{\partial a_j} - \frac{[B_i,B_j]}{a_i-a_j}\right)\frac{1}{z-a_i} - \left(\frac{\partial B_j}{\partial a_i} - \frac{[B_i,B_j]}{a_i-a_j}\right)\frac{1}{z-a_j} = 0.$$

Functions $1/(z-a_i)$ and $1/(z-a_j)$ define a basis if $i \neq j$. Hence,

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} = 0, & \\ \displaystyle\frac{\partial B_i}{\partial a_j} = \frac{[B_i,B_j]}{a_i-a_j}, \quad i \neq j, & \end{cases} \qquad i,j=\overline{1,n}. \tag{9}$$

Multiply the first equation of the system (9) by da_j and sum over all $i \neq j$:

$$dB_i - \frac{\partial B_i}{\partial a_i} da_j = -\sum_{j \neq i}^{n} \frac{[B_i,B_j]}{a_i-a_j}(-da_j).$$

Then we multiply the first equation of the system (9) by $(-da_i)$ and sum over all $i \neq j$:

$$-\sum_{j \neq i}^{n} \frac{\partial B_i}{\partial a_j} da_i = -\sum_{j \neq i}^{n} \frac{[B_i,B_j]}{a_i-a_j} da_i.$$

Add the obtained equations and take into account the second equation of the system (9):

$$dB_i = -\sum_{j \neq i}^{n} \frac{[B_i,B_j]}{a_i-a_j} d(a_i-a_j), \quad i=\overline{1,n}.$$

($\Leftarrow$)

Suppose that we have the following system

$$dB_i = -\sum_{j \neq i}^{n} \frac{[B_i,B_j]}{a_i-a_j} d(a_i-a_j), \quad i=\overline{1,n}.$$

Note that

$$\begin{aligned} dB_i(a) &= \frac{\partial B_i}{\partial a_1} da_1 + \cdots + \frac{\partial B_i}{\partial a_n} da_n \\ &= \sum_{j \neq i}^{n} \frac{\partial B_i}{\partial a_j} da_j + \frac{\partial B_i}{\partial a_i} da_i. \end{aligned}$$

Hence,

$$\sum_{j\neq i}^{n}\frac{\partial B_i}{\partial a_j}da_j+\frac{\partial B_i}{\partial a_i}da_i=\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j}da_j-\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j}da_i.$$

By comparing the corresponding coefficients at da_k, we have

$$\sum_{j\neq i}^{n}\frac{\partial B_i}{\partial a_j}=\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j},$$
$$\frac{\partial B_i}{\partial a_i}=-\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j}.$$

Add the obtained equations and take into account that the first equation holds for all $j\neq i$, we have

$$\begin{cases}\displaystyle\sum_{j=1}^{n}\frac{\partial B_i}{\partial a_j}=0, & \\ \displaystyle\frac{\partial B_i}{\partial a_j}=\frac{[B_i,B_j]}{a_i-a_j}, \quad j\neq i, & \end{cases} \qquad i=\overline{1,n}.$$

So, we have obtained the system (9) corresponding to the condition $d\omega_s=\omega_s\wedge\omega_s$. □

Theorem 3. *The system* (9) *is integrable.*

Proof. Let us consider the complex manifold $D\left(a^0\right)\times\underbrace{\mathfrak{gl}_p\left(\mathbb{C}\right)\times\cdots\times\mathfrak{gl}_p\left(\mathbb{C}\right)}_{n}$ with the coordinates $(a,B_1,\ldots,B_n)$ and np^2 equations on this space in the following form

$$dB_i+\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j}d\left(a_i-a_j\right)=0. \tag{10}$$

Let us consider the $1-$ form Ω_i

$$\Omega_i=dB_i+\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j}d\left(a_i-a_j\right).$$

If each Ω_i satisfies the conditions of Frobenius theorem, then for each set of initial data $B_1^0,\ldots,B_n^0$ there are holomorphic in $D\left(a^0\right)$ solutions $B_i\left(a\right)=B_i$, $B_i^0=B_i\left(a^0\right)$, of the equation (10).

$$d\Omega_i=d^2B_i+\sum_{j\neq i}^{n}d\left[B_i,B_j\right]\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}$$
$$=\sum_{j\neq i}^{n}\left([dB_i,B_j]+[B_i,dB_j]\right)\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}.$$

Take into account that $dB_i=-\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j}d\left(a_i-a_j\right)+\Omega_i$. Hence,

$$d\Omega_i=\sum_{j\neq i}^{n}\left(\left[\Omega_i-\sum_{j\neq i}^{n}\frac{[B_i,B_j]}{a_i-a_j}d\left(a_i-a_j\right),B_j\right]+\left[B_i,\Omega_j-\sum_{k\neq j}^{n}\frac{[B_j,B_k]}{a_j-a_k}d\left(a_j-a_k\right)\right]\right)\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}$$
$$=\sum_{j\neq i}^{n}\left[\Omega_i,B_j\right]\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}+\sum_{j\neq i}^{n}\sum_{k\neq i}^{n}\left[\left[B_i,B_k\right],B_j\right]\frac{d\left(a_i-a_j\right)\wedge d\left(a_i-a_k\right)}{\left(a_i-a_j\right)\left(a_i-a_k\right)}$$
$$+\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\left[B_i,\left[B_j,B_k\right]\right]\frac{d\left(a_i-a_j\right)\wedge d\left(a_j-a_k\right)}{\left(a_i-a_j\right)\left(a_j-a_k\right)}+\sum_{j\neq i}^{n}\left[B_i,\Omega_j\right]\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}.$$

Denote $\sum_{j\neq i}^{n}\sum_{k\neq i}^{n}\left[\left[B_i,B_k\right],B_j\right]\frac{d\left(a_i-a_j\right)\wedge d\left(a_i-a_k\right)}{\left(a_i-a_j\right)\left(a_i-a_k\right)}+\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\left[B_i,\left[B_j,B_k\right]\right]\frac{d\left(a_i-a_j\right)\wedge d\left(a_j-a_k\right)}{\left(a_i-a_j\right)\left(a_j-a_k\right)}$ by Θ_i and expand the coefficients at $\frac{d\left(a_i-a_j\right)}{a_i-a_j}$. Then

$$
\begin{aligned}
d\Omega_i &= \Theta_i+\sum_{j\neq i}^{n}\left(\Omega_iB_j-B_j\Omega_i+B_i\Omega_j-\Omega_jB_i\right)\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}\\
&= \Theta_i+\sum_{j\neq i}\left(\Omega_i\wedge B_j\frac{d\left(a_i-a_j\right)}{a_i-a_j}-\Omega_j\wedge B_i\frac{d\left(a_i-a_j\right)}{a_i-a_j}-B_j\Omega_i\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}+B_i\Omega_j\wedge\frac{d\left(a_i-a_j\right)}{a_i-a_j}\right).
\end{aligned}
$$

If $\Theta_i=0$, then the conditions of Frobenius theorem hold and we are done. So, by taking into account Jacobi identity $\left[\left[B_i,B_j\right],B_k\right]+\left[\left[B_j,B_k\right],B_i\right]+\left[\left[B_k,B_i\right],B_j\right]=0$, let us consider the first part of Θ_i:

$$
\sum_{j\neq i}^{n}\sum_{k\neq i}^{n}\left[\left[B_i,B_k\right],B_j\right]\frac{d\left(a_i-a_j\right)\wedge\left(a_i-a_k\right)}{\left(a_i-a_j\right)\left(a_i-a_k\right)}=\sum_{\substack{k<j\\ j,k\neq i}}^{n}\left(\left[\left[B_i,B_j\right],B_k\right]-\left[\left[B_i,B_k\right],B_j\right]\right)\frac{d\left(a_i-a_j\right)\wedge\left(a_i-a_k\right)}{\left(a_i-a_j\right)\left(a_i-a_k\right)}.
$$

By using this relation

$$
\left[\left[B_i,B_j\right],B_k\right]-\left[\left[B_i,B_k\right],B_j\right]=-\left[\left[B_j,B_k\right],B_i\right]\quad\left(=\left[B_i,\left[B_j,B_k\right]\right]\right),
$$

we obtain the following

$$
\begin{aligned}
\Theta_i &= \sum_{\substack{k<j\\ j,k\neq i}}^{n}\left[B_i,\left[B_j,B_k\right]\right]\frac{d\left(a_i-a_j\right)\wedge d\left(a_i-a_k\right)}{\left(a_i-a_j\right)\left(a_i-a_k\right)}+\sum_{j\neq k\neq i}^{n}\left[B_i,\left[B_j,B_k\right]\right]\frac{d\left(a_i-a_j\right)\wedge d\left(a_j-a_k\right)}{\left(a_i-a_j\right)\left(a_j-a_k\right)}\\
&= \sum_{\substack{k<j\\ j,k\neq i}}^{n}\left[B_i,\left[B_j,B_k\right]\right]\omega_1+\sum_{\substack{k<j\\ j\neq k\neq i}}^{n}\left[B_i,\left[B_j,B_k\right]\right]\omega_2+\sum_{\substack{k>j\\ j\neq k\neq i}}^{n}\left[B_i,\left[B_j,B_k\right]\right]\omega_2\\
&= \sum_{\substack{k<j\\ j,k\neq i}}^{n}\left[B_i,\left[B_j,B_k\right]\right]\omega_1+\sum_{\substack{k<j\\ j\neq k\neq i}}^{n}\left[B_i,\left[B_j,B_k\right]\right]\omega_2-\sum_{\substack{k<j\\ j\neq k\neq i}}^{n}\left[B_i,\left[B_j,B_k\right]\right]\omega_3,
\end{aligned}
$$

where $\omega_1=\frac{d\left(a_i-a_j\right)\wedge d\left(a_i-a_k\right)}{\left(a_i-a_j\right)\left(a_i-a_k\right)}$, $\omega_2=\frac{d\left(a_i-a_j\right)\wedge d\left(a_j-a_k\right)}{\left(a_i-a_j\right)\left(a_j-a_k\right)}$, and $\omega_3=\frac{d\left(a_i-a_k\right)\wedge d\left(a_j-a_k\right)}{\left(a_i-a_k\right)\left(a_j-a_k\right)}$. But

$$
\omega_1+\omega_2-\omega_3=0.
$$

Therefore, $\Theta_i\equiv 0$. □

3. Painlevé VI equation

We will start with the following

Example 3. The explicit solution of the given equation

$$
\frac{dy}{dt}=y^2,
$$

where y^2 is the holomorphic function in $\mathbb{C}$, is

$$
y\left(t\right)=-\frac{1}{t-t_0},
$$

and it has the simple pole at $t=t_0$, which depends on the t_0.

Definition 4. If singular points of a differential equation depend on the initial data, then such points are called *movable*.

Definition 5. A differential equation satisfies *Painlevé property*, if its explicit solution has movable simple poles only.

Remark (historic). Among equations of the form

$$\frac{du}{dt} = P(t,u),$$

where $P(t,u)$ is a meromorphic function in t and a rational function in u, the Riccati equation satisfies the Painlevé property only:

$$\frac{du}{dt} = a_0(t)\,u^2 + a_1(t)\,u + a_2(t),$$

where $a_i(t)$ are the meromorphic functions in t.

If we consider which equations of the following form

$$\frac{d^2u}{dt^2} = P\left(t,u,\frac{du}{dt}\right),$$

satisfy the Painlevé property, we find out that they are classified into 50 classes, and six of them define new special functions which are called *Painlevé transcendents.* The corresponding differential equations, that define these new function, are *Painlevé equations.*

We would like to show that *the sixth Painlevé equation*

$$\begin{aligned}\frac{d^2u}{dt^2} &= \frac{1}{2}\left(\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-t}\right)\left(\frac{du}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{u-t}\right)\frac{du}{dt} \\ &+ \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{u^2}+\gamma\frac{t-1}{(u-1)^2}+\delta\frac{t(t-1)}{(u-t)^2}\right)\end{aligned} \tag{11}$$

is equivalent to Schlesinger isomonodromic deformation of Fuchsian system of two differential equations with four simple poles.

So, let us consider the isomonodromic family of Fuchsian systems which consists of two differential equations and has four simple poles at the points 0, 1, t, and ∞:

$$\frac{dy}{dt} = B(z,t)\,y, \quad B(z,t) = \frac{B_0(t)}{z} + \frac{B_1(t)}{z-1} + \frac{B_2(t)}{z-t}, \tag{12}$$

where $t \in D\left(t^0\right) \subset \mathbb{C}\setminus\{0,1\}$.

Suppose that the monodromy group of the family (12) belongs to the Lie group $SL_2(\mathbb{C}) \ni \pi_1(\mathbb{C}\setminus\{0,1\},t_0)$, and residues $B_i(t) \in \mathfrak{sl}_2(\mathbb{C})$.

Demand that the residue $B_3 = -B_0 - B_1 - B_2$ at $t=\infty$ of the matrix $B(z,t)$ is diagonalizable.

Remark. If the system (12) is Schlesinger deformation, then B_3 is a constant matrix.

Denote the eigenvalues of $B_i(t)$ by $\pm\theta_i$.

Remark. The eigenvalues of the matrices B_i, $i = 0,1,2,3$, do not depend on the parameter t.

Since the matrix B_3 is diagonalizable, we can transform it to the diagonal matrix $B_3 = diag\{\theta_3,-\theta_3\}$. Then

$$\begin{aligned}B(z,t) = \frac{1}{z}&\begin{pmatrix} x_0 & -u(x_0-\theta_0) \\ u^{-1}(x_0+\theta_0) & -x_0\end{pmatrix} \\ &+ \frac{1}{z-1}\begin{pmatrix} x_1 & -v(x_1-\theta_1) \\ v^{-1}(x_1+\theta_1) & -x_1\end{pmatrix} \\ &\qquad + \frac{1}{z}\begin{pmatrix} x_2 & -w(x_2-\theta_2) \\ w^{-1}(x_2+\theta_2) & -x_2\end{pmatrix}.\end{aligned}$$

By taking in account that $B_3 = diag\{\theta_3,-\theta_3\}$, we have the following system of algebraic equations

$$\begin{gathered} x_0 + x_1 + x_2 = -\theta_3, \\ u(x_0-\theta_0) + v(x_1-\theta_1) + w(x_2-\theta_2) = 0, \\ u^{-1}(x_0+\theta_0) + v^{-1}(x_1+\theta_1) + w^{-1}(x_2-\theta_2) = 0. \end{gathered} \tag{13}$$

The element $\tilde{b}_{12}$ of the matrix B_3 equals zero, so the same element of the matrix $z(z-1)(z-t)B(z,t)$ is a polynomial in z of degree 1 at the fixed point t. Denote the unique root of this polynomial by $y(t)$.

Theorem 4. *If matrices $B_0(t)$, $B_1(t)$, $B_2(t)$ of the family (12) satisfy Schlesinger system (9), then the function $y(t)$ satisfies the sixth Painlevé equation (11), where the constants are*

$$\alpha = \frac{1}{2}\left(2\theta_3 - 1\right)^2, \quad \beta = -2\theta_0^2, \quad \gamma = 2\theta_1^2, \quad \delta = \frac{1}{2} - 2\theta_2^2.$$

Proof.

The first step.

Write down the element b_{12} of the matrix $B(z,t)$ in the following form

$$\begin{aligned} b_{12}(x,t) &= -\frac{u\left(x_0-\theta_0\right)}{z} - \frac{v\left(x_1-\theta_1\right)}{z-1} - \frac{w\left(x_2-\theta_2\right)}{z-t} \\ &= \frac{\varkappa\left(z-y(t)\right)}{z(z-1)(z-t)}, \end{aligned}$$

where $\varkappa = (t+1)\,u\left(x_0-\theta_0\right) + tv\left(x_1-\theta_1\right) + w\left(x_2-\theta_2\right)$ and $\varkappa y(t) = tu\left(x_0-\theta_0\right)$.

(14)
$$\begin{aligned} \varkappa &= (t+1)\,u\left(x_0-\theta_0\right) + tv\left(x_1-\theta_1\right) + w\left(x_2-\theta_2\right), \\ \varkappa y(t) &= tu\left(x_0-\theta_0\right). \end{aligned}$$

Denote

(15)
$$x(t) = \frac{x_0}{y} + \frac{x_1}{y-1} + \frac{x_2}{y-t}$$

by $x(t)$.

The equation (15) together with (13) and (14) defines six equations for nine functions x_0, x_1, x_2, u, v, w, x, and y of the parameter t. Express six of them thought last three functions. Hence, from the second equation of the system (13), we have

$$u = \frac{\varkappa y}{t\left(x_0-\theta_0\right)},$$

then the first equation is written as

$$\varkappa = \varkappa y + (t-1)\,v\left(x_1-\theta_1\right),$$

and we obtain

$$v = -\frac{\varkappa(y-1)}{(t-1)\left(x_1-\theta_1\right)}.$$

Substitute u and v into the second equation of the system (13). So,

$$w = \frac{\varkappa(y-t)}{t(t-1)\left(x_2-\theta_2\right)}.$$

Substitute found functions u, v, w into the third equation of the system (13). Then we have got the system of algebraic equations with respect to x_0, x_1, and x_2:

$$\begin{aligned} x_0 + x_1 + x_3 &= -\theta_3, \\ \frac{x_0}{y} + \frac{x_1}{y-1} + \frac{x_2}{y-t} &= x, \\ \frac{t}{y}\left(x_0^2-\theta_0^2\right) - \frac{t-1}{y-1}\left(x_1^2-\theta_1^2\right) + \frac{t(t-1)}{y-t}\left(x_2^2-\theta_2^2\right) &= 0. \end{aligned}$$

The obtained system has the following solution

(16)
$$\begin{aligned} x_0 &= \frac{y}{2\theta_3 t}\left(y(y-1)(y-t)x^2 + 2\theta_3(y-1)(y-t)x - \theta_0^2\frac{t}{y} + \theta_1^2\frac{t-1}{y-1} - \theta_2^2\frac{t(t-1)}{y-t} + (y-t-1)\,\theta_3^2\right), \\ x_1 &= -\frac{y-1}{2\theta_3(t-1)}\left(y(y-1)(y-t)x^2 + 2\theta_3 y(y-t)x - \theta_0^2\frac{t}{y} + \theta_1^2\frac{t-1}{y-1} - \theta_2^2\frac{t(t-1)}{y-t} + (y-t+1)\,\theta_3^2\right), \\ x_2 &= \frac{y-t}{2\theta_3 t(t-1)}\left(y(y-1)(y-t)x^2 + 2\theta_3(y-1)x - \theta_0^2\frac{t}{y} + \theta_1^2\frac{t-1}{y-1} - \theta_2^2\frac{t(t-1)}{y-t} + (y+t-1)\,\theta_3^2\right). \end{aligned}$$

Therefore, we have got the expressions of the functions x_0, x_1, x_2, u, v, and w via $\varkappa$, x, and y.

The second step.
Recall that the family (12) is Schlesinger isomonodromic deformation, i.e. the differential 1 – form

$$\omega_s = \sum_{i=0}^{2} \frac{B_i(a)}{z-a_i} d(z-a_i)$$

satisfies Frobenius integrability condition

$$d\omega_s = \omega_s \wedge \omega_s.$$

Write down this condition for the matrix $B(z,t)$

$$\frac{\partial B(z,t)}{\partial t} = \frac{B_2(t)}{(z-t)^2} + \frac{[B(z,t), B_2(t)]}{z-t},$$

or for the matrix elements of $B(z,t)$

$$\frac{\partial b_{11}}{\partial t} = \frac{x_2}{(z-t)^2} + \frac{1}{z-t}\left(b_{21} w (x_2-\theta_2) + b_{21} w^{-1}(x_2+\theta_2)\right), \tag{17}$$
$$\frac{\partial b_{12}}{\partial t} = -\frac{w(x_2-\theta_2)}{(z-t)^2} + \frac{1}{z-t}\left(-2b_{12}x_2 - 2b_{11} w (x_2-\theta_2)\right).$$

On the other hand, we can take the derivative of $b_{12} = \dfrac{\varkappa(z-y)}{z(z-1)(z-t)}$ with respect to t:

$$\frac{\partial b_{12}}{\partial t} = \frac{\frac{d\varkappa}{dt}(z-y)(z-t) - \varkappa \frac{dy}{dt}(z-t) + \varkappa(z-y)}{z(z-1)(z-t)^2}. \tag{18}$$

By comparing relations of $\partial_t b_{12}$ from (17) and (18), we are able to obtain the differential relations of the functions $x(t)$, $y(t)$, and $\varkappa(t)$:

$$\frac{1}{\varkappa}\frac{d\varkappa}{dt} - \frac{1}{z-y}\frac{dy}{dt} + \frac{1}{z-t} = -\frac{z(z-1)(z-t)}{t(t-1)(z-t)(z-y)} - \frac{2x_2}{z-t} - 2\left(\frac{x_0}{z} + \frac{x_1}{z-1} + \frac{x_2}{z-t}\right)\frac{z(z-1)(y-t)}{t(t-1)(z-y)}.$$

Resides at $z=0$ and $z=1$ are

$$\frac{1}{\varkappa}\frac{d\varkappa}{dt} + \frac{1}{y}\frac{dy}{dt} - \frac{1}{t} = \frac{2x_2}{t} - 2x_0 \frac{y-t}{t(t-1)y}, \tag{19}$$
$$\frac{1}{\varkappa}\frac{d\varkappa}{dt} - \frac{1}{1-y}\frac{dy}{dt} + \frac{1}{1-t} = \frac{2x_2}{t-1} - 2x_1 \frac{y-t}{t(t-1)(1-y)}$$

respectively.

Furthermore, we find out an expression of $\dfrac{dx}{dt}$, if we represent the function $x(t)$ in the form $x(t) = b_{11}(y,(t),t)$. Then, by taking into account the first equation of the system (17) and the fact that $b_{12}(y(t),t) = 0$, we get

$$\frac{dx}{dt} = \left.\left(\frac{\partial b_{11}}{\partial z}\frac{dy}{dt} + \frac{\partial b_{11}}{\partial t}\right)\right|_{z=y(t)}$$
$$= -\left(\frac{x_0}{y} + \frac{x_1}{(y-1)^2} + \frac{x_2}{(y-t)^2}\right)\frac{dy}{dt} + \frac{x_2}{(y-t)^2} + \frac{1}{y-t} b_{21}(y,t) w (x_2-\theta_2).$$

Since

$$b_{21}(y,t) = \frac{x_0+\theta_0}{yu} + \frac{x_1+\theta_1}{(y-1)v} + \frac{x_2+\theta_2}{(y-t)w}$$
$$= \frac{t}{\varkappa y^2}\left(x_0^2-\theta_0^2\right) - \frac{t-1}{\varkappa(y-1)^2}\left(x_1^2-\theta_1^2\right) + \frac{t(t-1)}{(y-t)^2}\left(x_2^2-\theta_2^2\right),$$

then

$$b_{21}(y,t)\frac{w(x_2-\theta_2)}{y-t} = b_{21}(y,t)\frac{\varkappa}{t-1}$$
$$= \frac{1}{t(t-1)}\left(\frac{t}{y^2}\left(x_0^2-\theta_0^2\right) - \frac{t-1}{(y-1)^2}\left(x_1^2-\theta_1^2\right) + \frac{t(t-1)}{(y-t)^2}\left(x_2^2-\theta_2^2\right)\right).$$

Therefore, the derivative of x with respect to t is expressed in the following way

$$\frac{dx}{dt} = -\left(\frac{x_0}{y^2} + \frac{x_1}{(y-1)^2} + \frac{x_2}{(y-t)^2}\right)\frac{dy}{dt} + \frac{x_2}{(y-t)^2}$$
$$+ \frac{1}{t(t-1)}\left(\frac{t}{y^2}\left(x_0^2 - \theta_0^2\right) - \frac{t-1}{(y-1)^2}\left(x_1^2 - \theta_1^2\right) + \frac{t(t-1)}{(y-t)^2}\left(x_2^2 - \theta_2^2\right)\right).$$

If we substitute expressions of the functions x_0, x_1, and x_2 (16) into the obtained result for $\frac{dx}{dt}$ and then use the subtraction of the expressions (19), we obtain the system of differential equations of the first order with respect to $x(t)$ and $y(t)$:

(20)

$$\frac{dx}{dt} = \frac{1}{t(t-1)}\left(\left(2ty - 3y^2 + 2y - t\right)x^2 + (1-2y)x - \theta_0^2\frac{t}{y^2} + \theta_1^2\frac{t-1}{(y-1)^2} - \theta_2^2\frac{t(t-1)}{(y-t)^2} + \theta_3(\theta_3 - 1)\right),$$
$$x = \frac{1}{2}\left(\frac{t(t-1)}{y(y-1)(y-t)}\frac{dy}{dt} - \frac{1}{y-t}\right).$$

By substitution $x(t)$ into the first equation of the system (20), we obtain the equation (11) with respect to $y(t)$ whose parameters are determined by the theorem condition. □

References

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