AN OVERVIEW ON KODAIRA DIMENSION OF THE MODULI SPACE OF PPAV

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In the theory of the classification of algebraic varieties a fundamental birational invariant is the Kodaira dimension. Let X be smooth projective defined over $\mathbb{C}$, we can consider the space of sections of the canonical bundle K_X and of its powers, mK_X. When X is not smooth, neither projective, we can consider a smooth completion of X, in fact these spaces of sections are birational invariants. Thus we can consider rational maps (i.e. they are not necessarily definied everywhere)

$$\Psi_m : X \to \mathbf{P}^n(\mathbb{C})$$

We can say that the variety X is of general type if , for $m >> 0$ the map Ψ_m is birational onto its image. More generally, the Kodaira dimension of is defined as the dimension of the image. . The Kodaira dimension is a birational invariant, that is, it does not depend on the representative in the birational equivalence class. To the opposite of the varieties of general type we have the case in which $|mK_X| = \emptyset$, i.e the Kodaira dimension is $-\infty$. Among these varieties, an important role is played by the unirational varieties, i.e. those for which there exists a dominant rational morphism $\mathbf{P}^n \to X$.

We will report on the case of the moduli space of principally polarized abelian varieties $\mathcal{A}_n$,i.e of the pairs (A, Θ) with A an abelian variety and Θ a principal polarization. In the complex case we have an explicit realization of these moduli spaces as $\mathcal{A}_n = \mathbb{H}_n/Sp(2n, \mathbb{Z})$, here $\mathbb{H}_n$ is the Siegel upper half space of degree n.

The problem of unirationality and Kodaira dimension of $\mathcal{A}_n$ is related to modular forms of weight k, i.e to holomorphic functions defined on $\mathbb{H}_n$ such that

$$f(M \cdot \tau) = det(c\tau + d)^k f(\tau) \text{ for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2n, \mathbb{Z}).$$

In fact, sections of $mK_{\mathcal{A}_n}$ are of the form

$$\omega = f(\tau)(d\tau)^{\otimes m},\ d\tau = d\tau_{11} \wedge d\tau_{12} \wedge \cdots \wedge d\tau_{nn}$$

and f a modular form of weight $m(n + 1)$. Modular forms have Fourier expansion

$$f(\tau) = \sum_{T even\, integral \geq 0} a(T) exp(\pi i tr(T\tau)).$$

A modular form is a cusp form if $a(T) \neq 0$ implies $T > 0$. The vanishing along the boundary of a cusp form f is given by

$$b = \frac{1}{2} min T[x]$$

Here the minimum is among all T such that $a(T) \neq 0$ and $x \in \mathbb{Z}^n \setminus 0$.

Let $\mathcal{A}_n^0$ be the set of smooth points of $\mathcal{A}_n$, thus the main problem is the extension of these forms to a smooth compactification. There are two kinds of obstructions produced by resolution of singularities and compactification. As first result, at the begin of seventies, Freitag in a sequence of papers proved that a canonical differential forms extends providing that the modular form f is a cusp form.

Hence he was able to prove that $\mathcal{A}_n$ is not unirational for $n \equiv 0 \; mod\, 24$,[4]. Moreover, at the same time, he proved the not unirationality of $\mathcal{A}_n$ for $n \equiv 1 \; mod\, 8, n \geq 17$, [5], constructing explicitely sections of the sheaf of holomorphic differential forms, not of maximal degree. These results induced to the development of the study of modular forms with pluriharmonic coefficients and of vector valued modular forms.

At the end of seventies in a milestone paper [9], Tai gave a criterion on the extensibility of pluri-canonical differential forms and proved that there are sufficiently many pluri-canonical forms which extend for $n \geq 9$, so that in these cases $\mathcal{A}_n$ is of general type. After that there have been extension of this result to the case $n \geq 8$, [6] and to $n \geq 7$, [8]. Meanwhile several authors gave proofs about the unirationality of $\mathcal{A}_n$ for $n \leq 5$, [1],[3],[7],[10]. Thus the only case that remained open whose the case $n = 6$. Recently, in [2], inspired by Freitag's method, Dittmann, Scheithauer and I gave explicitely the existence of a bicanonical differential forms on $\mathcal{A}_6$ proving the not unirationality of this space. This results to be related to a theta series with harmonic polynomial coefficients related to the Leech's lattice Λ_{24}.Hence at the moment, as definitive result , we have that $\mathcal{A}_n$ is unirational if and only if $n \leq 5$.

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