Whitham Equations and the Chazy Equation

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Integrable Systems and Automorphic Forms

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The Heat Equation

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Differentiating this equation with respect to x, we obtain the Burgers equation

$$k_t = k_{xx} + 2kk_x.$$

Now we introduce a new function l such that $l = -k_x$. Differentiating the Burgers equation with respect to x, we obtain

$$l_t = l_{xx} - 2l^2 + 2kl_x.$$

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The Ansatz

Now we are looking for the function $Q(I, t)$ such that

$$I_x = Q.$$

Then we obtain two equations (where $I_{xx} = QQ'$. Here $Q' \equiv \partial Q/\partial I$)

$$I_x = Q, \quad I_t = QQ' - 2I^2 + 2kQ.$$

The Ansatz

Now we are looking for the function $Q(l,t)$ such that

$$l_x = Q.$$

Then we obtain two equations (where $l_{xx} = QQ'$. Here $Q' \equiv \partial Q/\partial l$)

$$l_x = Q, \quad l_t = QQ' - 2l^2 + 2kQ.$$

Their compatibility condition $(l_t)_x = (l_x)_t$ leads to

$$Q_t = Q^2 Q'' + 2l^2 Q' - 6lQ.$$

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Now we introduce a new function $G(I, t) = Q^2$. Then we obtain

$$G_t = GG'' - \frac{1}{2}G'^2 + 2I^2 G' - 12IG.$$

The Darboux–Halphen system

Now we are looking for a particular solution of this equation

$$G_t = GG'' - \frac{1}{2}G'^2 + 2l^2G' - 12lG,$$

in the ansatz

$$G = 4(l - l_1)(l - l_2)(l - l_3),$$

where $l_k(t)$ are some functions.

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$$\frac{1}{2}l_1' = l_2 l_3 - l_1(l_2 + l_3),$$

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The above ansatz

$$G = 4(l - l_1)(l - l_2)(l - l_3),$$

we can rewrite in the expanded form

$$G = 4l^3 + al^2 + bl + c,$$

where

$$a = -4(l_1 + l_2 + l_3), \quad b = 4(l_1 l_2 + l_1 l_3 + l_2 l_3), \quad c = -4 l_1 l_2 l_3.$$

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Then we obtain the remarkable Chazy equation

$$a''' = 2aa'' - 3(a')^2,$$

where

$$b = \frac{1}{2} a', \quad c = \frac{1}{24} a''.$$

The Intermediate Resume

The linear heat equation $\theta_t = \theta_{xx}$ possesses a particular solution, such that

$$\theta_x = k\theta, \quad k_x = -l, \quad l_x = \sqrt{4l^3 + al^2 + \frac{1}{2}a'l + \frac{1}{24}a''},$$

$$\theta_t = (k^2 - l)\theta, \quad k_t = -l_x - 2kl,$$

$$l_t = 4l^2 + al + \frac{1}{4}a' + 2k\sqrt{4l^3 + al^2 + \frac{1}{2}a'l + \frac{1}{24}a''},$$

where the function $a(t)$ is a general solution of the Chazy equation

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Here $\theta(x, t)$ is a famous Jacobi theta-function, while its second logarithmic derivative $l = -\partial_x^2 \ln\theta$ is an elliptic function

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where the functions $l_k(t)$ solve the Darboux–Halphen system

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$$W_n \equiv < w^n > = \frac{1}{T} \oint w^n d\theta,$$

where period

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is fixed. For instance $T = 2\pi$. Then we obtain

$$(w^n w')' = w^n w'' + n w^{n-1} w'^2 = (6+4n) w^{n+2} - \left(\frac{1}{2} + n\right) g_2 w^n - g_3 n w^{n-1},$$

where we used the differential consequence

$$w'' = 6w^2 - \frac{1}{2} g_2.$$

The Classical Recursive Set

The averaging of this set

$$(w^n w')' = (6+4n)w^{n+2} - \left(\frac{1}{2}+n\right) g_2 w^n - g_3 n w^{n-1},$$

lead to an infinite recursive set of the averaged expressions

$$(6+4n)W_{n+2} - \left(\frac{1}{2}+n\right) g_2 W_n - g_3 n W_{n-1} = 0,$$

where we used

$$W_n \equiv < w^n > = \frac{1}{T} \oint w^n d\theta,$$

with the fixed period

$$T = \oint d\theta = \oint \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}}.$$

The Classical Recursive Set

The above infinite recursive set of the averaged expressions

$$(6+4n)W_{n+2} - \left(\frac{1}{2}+n\right) g_2 W_n - g_3 n W_{n-1} = 0,$$

can be written in the following unique form

$$(2n+3)W_{n+2} = 3(2n+1)W_2 W_n + n(5W_3 - 9W_1 W_2)W_{n-1},$$

where

$$g_2 = 12W_2, \quad g_3 = 10W_3 - 18W_1 W_2.$$

The Bogaevsky approach

V.N. Bogaevsky made the following observation: if the function w will be differentiated by any extra parameter γ, then we can obtain another set of differential recursive relationships

$$24W_2W'_{n+1} - 6(2n+1)W_{n+1}W'_2 + 6[5W_3 - 9W_1W_2]W'_n$$

$$+(1-2n)W_n[5W_3 - 9W_1W_2]' = 0.$$

In our case of the elliptic curve we can restrict our consideration when $n = 0$ and $n = 1$

$$W'_3 = 3\gamma W'_2 - 3W_2, \quad 5W_3 = 5\gamma W_2 + (\gamma^2 - W_2)W'_2,$$

where we put $\gamma \equiv W_1$.

Then this system (we remind that $\gamma \equiv W_1$)

$$W_3' = 3\gamma W_2' - 3W_2, \quad 5W_3 = 5\gamma W_2 + (\gamma^2 - W_2)W_2',$$

can be written in a little bit more simple form

$$\beta' = \alpha, \quad 5\beta = \alpha\alpha' + 2\gamma\alpha,$$

where auxiliary modulation parameters

$$\alpha = \frac{1}{6} < (w - W_1)^2 > , \qquad \beta = -\frac{1}{36} < (w - W_1)^3 >$$

are proportional to mean-square and mean-cube amplitudes w, respectively.

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Then this system (we remind that $\gamma \equiv W_1$)

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Eliminating β we obtain the equation of the second order

$$\alpha\alpha'' + \alpha'^2 + 2\gamma\alpha' - 3\alpha = 0.$$

The Chazy Equation

This equation (here we remind that the independent variable is γ)

$$\alpha\alpha'' + \alpha'^2 + 2\gamma\alpha' - 3\alpha = 0$$

is connected with the Chazy equation (here we remind that the independent variable is t)

$$\gamma''' = 2\gamma\gamma'' - 3(\gamma')^2$$

by the transformation

$$\gamma'' = \alpha\frac{d\alpha}{d\gamma}, \quad \gamma''' = \alpha^2\frac{d^2\alpha}{d\gamma^2} + \alpha(\frac{d\alpha}{d\gamma})^2,$$

where we denote $\alpha = -\gamma'$.