Jacobi forms and root systems

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1 Jacobi forms I

Let L be the lattice generated by a root system R of rank N with Weyl group W.

Definition. A weak Weyl-invariant Jacobi form of weight k and index m ($k \in \mathbb{Z}$, $m \in \mathbb{N}$) for a root lattice L with an inner product $(\cdot, \cdot)$ is a holomorphic function $\varphi : \mathcal{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$, such that:

1) $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^k e^{\pi i m \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}} \varphi(\tau, \mathfrak{z})$;

2) $\varphi(\tau, \mathfrak{z} + l\tau + l') = e^{-2\pi i m(l,\mathfrak{z}) - \pi i m(l,l)\tau} \varphi(\tau, \mathfrak{z})$ for $l, l' \in L$;

3) $\varphi(\tau, w(\mathfrak{z})) = \varphi(\tau, \mathfrak{z})$ for w from Weyl group W;

4) $\varphi(\tau, \mathfrak{z})$ has a Fourier expansion:

$$\sum_{\lambda \in L^*, n \geqslant 0} a(n, l) q^n \zeta^\lambda,$$

where $q = e^{2\pi i \tau}$, $\zeta^\lambda = e^{2\pi i (\mathfrak{z}, \lambda)}$.

2 Chevalley type theorems

Theorem (Chevalley-Shephard-Todd) Let W be the group generated be a Coxeter group acting on the space V $(\dim V = N)$. Then

$$S^N(V_{\mathbb{C}})^W \simeq \mathbb{C}[x_1,\ldots,x_N]^W \simeq \mathbb{C}[y_1,\ldots,y_N],$$

where $y_1,\ldots,y_N$ are homogeneous W-invariant polynomials.

Theorem (E. Looijenga, I. Bernstein & O. Schwarzman, V. Kac & D. Peterson) Let W be a complex crystallographic Coxeter group acting on a complex affine space of dimension N. Then the algebra of invariants is a polynomial algebra generated by $N+1$ theta-functions.

Theorem (K. Wirtmüller). For every root system R (except E_8) the set of all weak Weyl-invariant Jacobi forms $J^W_{*,*}(R)$ has the structure of a free bigraded algebra over the ring of modular forms with $n+1$ generators

$$J^{w,W}_{*,*}(R) = M_*[\varphi_{0,1}, \varphi_{-k(1),m(1)}, \ldots, \varphi_{-k(n),m(n)}], j = 1,\ldots,n,$$

where the set of all $k(j)$ is the set of degrees of $W(R)$-invariant polynomials and the set of all $m(j)$ is the set of coefficients of the highest coroot $\tilde{\alpha}$ written as a linear combination of basis elements of R.

3 About Wirthmüller theorem

- K. Wirthmüller, *Root systems and Jacobi forms.* Comp. Math. **82** (1992), 293–354.

- M. Bertola, *Frobenius manifold structure on orbit space of Jacobi group*, PhD thesis (2000).

- H. Wang, *Weyl invariant E_8 Jacobi forms.* arXiv:1801.08462.

- D.A., V. Gritsenko, *The D_8-tower of weak Jacobi forms and applications.* J. Geom. Phys., electronically published on February 6, 2020, DOI: https://doi.org/10.1016/j.geomphys.2020.103616 (to appear in print).

- In preparation: D.A. about F_4 and D.A., V. Gritsenko about $C_n \& D_n$.

Theorem in case D_n. For root systems of D_n type the set of all weak Weyl-invariant Jacobi forms is a free algebra over the ring of modular forms and

$$J^W_{*,*}(D_2) = M_*[\varphi_{0,1}, \varphi_{-2,1}, \varphi_{-4,1}],$$

$$J^W_{*,*}(D_3) = M_*[\varphi_{0,1}, \varphi_{-2,1}, \varphi_{-4,1}, \omega_{-3,1}],$$

$$J^W_{*,*}(D_4) = M_*[\varphi_{0,1}, \varphi_{-2,1}, \varphi_{-4,1}, \varphi_{-6,2}, \omega_{-4,1}],$$

$$J^W_{*,*}(D_n) = M_*[\varphi_{0,1}, \varphi_{-2,1}, \varphi_{-4,1}, \varphi_{-6,2}, \ldots, \varphi_{-2n+2,2}, \omega_{-n,1}].$$

Theorem in case C_n. For root systems of C_n type the set of all weak Weyl-invariant Jacobi forms is a free algebra over the ring of modular forms and

$$J^W_{*,*}(C_n) = M_*[\varphi_{0,1}, \varphi_{-2,1}, \varphi_{-4,1}, \varphi_{-6,2}, \ldots, \varphi_{-2n+2,2}, \omega^2_{-n,1}].$$

4 Construction of generators

$$\omega_{-n,1}^{D_n}(\tau, \mathfrak{z}) = \frac{\vartheta_{11}(\tau, z_1) \cdot \ldots \cdot \vartheta_{11}(\tau, z_n)}{\eta^{3n}(\tau)}.$$

Properties:

- anti-invariant under change of sign for an odd number of $\mathfrak{z}$-coordinates,
- restriction of this form to D_{n-1} gives 0,
- well-known divisor.

Construction of $\varphi_{0,1}^{D_n}$. The first way

For any Jacobi form one can define Hecke operator $T_-(t)$ that increase the index of Jacobi form in t times, doesn't change the weight and doesn't decrease the divisor of the form. More precisely, for $\varphi_{k,m} \in J_{k,m}(L)$

$$\left(\varphi_{k,m}\Big|_{T_-(t)}\right)(\tau, \mathfrak{z}) := \sum_{\substack{ad=t \\ b \bmod d}} a^k t^{-1} \varphi_{k,m}\left(\frac{a\tau + b}{d}, a\mathfrak{z}\right).$$

In our case we obtain the well-defined form

$$\varphi_{0,1}^{D_n} = 2\frac{\omega_{-n,1}\Big|_{T_-(2)}}{\omega_{-n,1}} =$$

$$= 2^{-n}\frac{\omega_{-n,1}(2\tau, 2\mathfrak{z})}{\omega_{-n,1}(\tau, \mathfrak{z})} + \frac{\omega_{-n,1}(\frac{\tau}{2}, \mathfrak{z})}{\omega_{-n,1}(\tau, \mathfrak{z})} + \frac{\omega_{-n,1}(\frac{\tau+1}{2}, \mathfrak{z})}{\omega_{-n,1}(\tau, \mathfrak{z})}.$$

Construction of $\varphi_{0,1}^{D_n}$. The second way

For another way to construct $\varphi_{0,1}^{D_n}$ we can consider

$$\varphi_{0,1} = \frac{\vartheta_{00}(\tau, z_1) \cdot \ldots \cdot \vartheta_{00}(\tau, z_n)}{\vartheta_{00}^n(\tau, 0)} + \frac{\vartheta_{01}(\tau, z_1) \cdot \ldots \cdot \vartheta_{01}(\tau, z_n)}{\vartheta_{01}^n(\tau, 0)} +$$

$$+ \frac{\vartheta_{10}(\tau, z_1) \cdot \ldots \cdot \vartheta_{10}(\tau, z_n)}{\vartheta_{10}^n(\tau, 0)}.$$

It follows from the properties of even theta-functions that constructed form has weight 0 and index 1 for D_n. However, there are also the following relations:

$$\frac{\vartheta_{00}(\tau, z)}{\vartheta_{00}(\tau, 0)} = \frac{\vartheta_{11}(\frac{\tau+1}{2}, z)}{\vartheta_{11}(\tau, z)} \cdot \frac{\eta^3(\tau)}{\eta^3(\frac{\tau+1}{2})};$$

$$\frac{\vartheta_{01}(\tau, z)}{\vartheta_{01}(\tau, 0)} = \frac{\vartheta_{11}(\frac{\tau}{2}, z)}{\vartheta_{11}(\tau, z)} \cdot \frac{\eta^3(\tau)}{\eta^3(\frac{\tau}{2})};$$

$$\frac{\vartheta_{10}(\tau, z)}{\vartheta_{10}(\tau, 0)} = \frac{1}{2} \frac{\vartheta_{11}(2\tau, 2z)}{\vartheta_{11}(\tau, z)} \cdot \frac{\eta^3(\tau)}{\eta^3(2\tau)}.$$

Differential operator

Definition. Let L be a positive definite lattice of rank n with inner product $(\cdot,\cdot)$.And let $\varphi_{k,m}(\tau,\mathfrak{z})$ be a Jacobi forms of weight k and index m. Then one can defined a modular differential operator

$$H_k(\varphi_{k,m}(\tau,\mathfrak{z})) =$$

$$= \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\varphi_{k,m}(\tau,\mathfrak{z}) + \frac{1}{8\pi^2 m}\left(\frac{\partial}{\partial\mathfrak{z}},\frac{\partial}{\partial\mathfrak{z}}\right)\varphi_{k,m}(\tau,\mathfrak{z}) + (n-2k)G_2(\tau)\varphi_{k,m}(\tau,\mathfrak{z}),$$

where $G_2(\tau) = -\frac{1}{24} + q\cdot(\ldots)$ is a quasimodular Eisenstein series of weight 2.

Using Fourier development of Jacobi form $\varphi_{k,m}(\tau,\mathfrak{z})$ one can obtain another definition of action H_k on $\varphi_{k,m}(\tau,\mathfrak{z})$:

$$H_k(\varphi_{k,m}(\tau,\mathfrak{z})) = \sum_{n\geqslant 0, l\in L^\vee}(2n - \frac{1}{m}(l,l))a(n,l)q^n\zeta^l + (n-2k)G_2(\tau)\varphi_{k,m}(\tau,\mathfrak{z}).$$

This form is more convenient for calculations.

Construction of $\varphi_{-4,1}^{D_n}$

Applying differential operators and multiplication by Eisenstein series to $\varphi_{0,1}^{D_n}$ we can obtain following forms:

Weight 2	Weight 4	Weight 6	Weight 8
$H_0(\varphi_{0,1})$	$H_2(H_0(\varphi_{0,1}))$	$H_4(H_2(H_0(\varphi_{0,1})))$	$H_6(H_4(H_2(H_0(\varphi_{0,1}))))$
	$E_4\varphi_{0,1}$	$H_4(E_4\varphi_{0,1})$	$H_6(H_4(E_4\varphi_{0,1}))$
		$E_4H_0(\varphi_{0,1})$	$H_6(E_4H_0(\varphi_{0,1}))$
		$E_6\varphi_{0,1}$	$H_6(E_6\varphi_{0,1})$
			$E_4H_2(H_0(\varphi_{0,1}))$
			$E_6H_0(\varphi_{0,1})$
			$E_4^2\varphi_{0,1}$

Construction of $\varphi_{-4,1}^{D_n}$

Direct computations give us the following linear combination:

$$\varphi_{8,1}^{D_n} =$$

$$= H_6(E_4 H_0(\varphi_{0,1})) - H_6(E_6\varphi_{0,1}) - (n+6)E_4^2\varphi_{0,1} + 2(n+6)E_6 H_0(\varphi_{0,1}) =$$

$$= q\,(\ldots)$$

As a result we obtain (up to multiplication by non-zero constant)

$$\varphi_{-4,1}^{D_n} = \frac{\varphi_{8,1}^{D_n}}{\Delta}.$$

Construction of $\varphi_{-2,1}^{D_n}$

Now, applying the differential operator to $\varphi_{-4,1}^{D_n}$, we can finally construct the last form of index 1:

$$\varphi_{-2,1}^{D_n} = H_{-4,1}(\varphi_{-4,1}^{D_n}).$$

Forms of index 2

Let us notice that

$$D_n < \mathbb{Z}^n.$$

Consequently,

$$D_n(2) < (\mathbb{Z}(2))^{\oplus n} \simeq (A_1)^{\oplus n}.$$

Hence

$$\varphi^{D_n}_{-2k,2} = \frac{1}{k!(n-k)!} \sum_{\sigma \in S_n} \phi_{-2,1}(\tau, z_{\sigma(1)}) \dots \phi_{-2,1}(\tau, z_{\sigma(k)}) \times \\ \times \phi_{0,1}(\tau, z_{\sigma(k+1)}) \dots \phi_{0,1}(\tau, z_{\sigma(n)}) \in J^W_{-2k,2}(D_n),$$

Differential equations

Examples of differential equations:

- $H_{-n}(\omega^{D_n}_{-n,1}) = 0$;
- for any α and β in weight 6:

$$\alpha H_4(H_2(H_0(\varphi_{0,1}))) + \beta H_4(E_4\varphi_{0,1}) + \gamma E_4 H_0(\varphi_{0,1}) + \delta E_6 \varphi_{0,1} = q \cdot (\ldots),$$

where

$$\gamma = -\frac{1}{3}\left((5n^2+18n+16)\alpha + 3n\beta\right), \delta = \frac{1}{3}\left((n^2+18n+32)\alpha + 12\beta\right).$$

Consequently, we can obtain Jacobi form of weight -6 and index 1 for D_n. But it is impossible by the main theorem.

5 Jacobi group

Let L a positive definite lattice with inner product $(\cdot\,,\cdot)$. **Heisenberg group** $H(L)$ for this lattice is a central extension $(L\times L)\ltimes\mathbb{Z}$:

$$H(L)=\{\mathbf{h}=[\lambda,\mu,r]:\lambda,\mu\in L, r\in\frac{1}{2}\mathbb{Z}\quad\text{with}\quad r+\frac{1}{2}(\lambda,\mu)\in\mathbb{Z}\},$$

$$\mathbf{h}\cdot\mathbf{h'}=[\lambda,\mu,r]\cdot[\lambda',\mu',r']=[\lambda+\lambda',\mu+\mu',r+r'+\frac{1}{2}((\lambda,\mu')-(\lambda',\mu))].$$

If L is a root lattice, one can add the action of Weyl group

$$\mathcal{W}=W\ltimes H(L):$$

$$(w,\mathbf{h})\cdot(w',\mathbf{h'})=(w\cdot w',[\lambda+w\lambda',\mu+w\mu',r+r'+\frac{1}{2}((\lambda,w\mu')-(w\lambda',\mu))]).$$

Finally, **Jacobi group** is

$$J(L) = SL_2(\mathbb{Z}) \ltimes \mathcal{W} = SL_2(\mathbb{Z}) \ltimes (W \ltimes H(L))$$

with action on $\mathcal{W}$ by

$$\gamma.w = w, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$$\gamma.\mathbf{h} = [d\lambda - c\mu, -b\lambda + a\mu, r]$$

$$(\gamma, w, \mathbf{h}) \cdot (\gamma', w', \mathbf{h'}) = (\gamma \cdot \gamma', w \cdot w', \mathbf{h} \cdot (\gamma.\mathbf{h'})).$$

This group is acting on **the tube domain**

$$\mathcal{H}(L) = \mathcal{H} \oplus (L \otimes \mathbb{C}) \oplus \mathcal{H} = \{(\tau, \mathfrak{z}, \omega), 2\mathrm{Im}(\tau)\mathrm{Im}(\omega) - (\mathrm{Im}(\mathfrak{z}), \mathrm{Im}(\mathfrak{z})) > 0\}$$

$$\gamma(\tau, \mathfrak{z}, u) = \left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}, \omega - \frac{c(\mathfrak{z}, \mathfrak{z})}{2(c\tau + d)} \right);$$

$$w(\tau, \mathfrak{z}, u) = (\tau, w(\mathfrak{z}), \omega);$$

$$\mathbf{h}(\tau, \mathfrak{z}, u) = \left(\tau, \mathfrak{z} + \lambda\tau + \mu, \omega + \frac{(\lambda, \lambda)}{2}\tau + (\lambda, \mathfrak{z}) + \frac{(\lambda, \mu)}{2} + r \right).$$

6 Jacobi forms II

Let L be the lattice generated by root system R of rank N with Weyl group W, and $J(M, L)$ equals to $(c\tau + d)$ for $\gamma \in SL_2(\mathbb{Z})$ and 1 for $[\lambda, \mu, r] \in H(L)$.

Definition. A weak Weyl-invariant Jacobi form of weight k and index m (k, $m \in \mathbb{Z}$) for a root lattice L with an inner product $(\cdot, \cdot)$ is a holomorphic function $\varphi : \mathcal{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$, if the function

$$\tilde{\varphi}(Z) = \varphi(\tau, \mathfrak{z}) e^{2\pi i m \omega}, Z \in \mathcal{H}(L)$$

transforms like

$$\tilde{\varphi}(M(Z)) = J(M, Z)^{-k} \tilde{\varphi}(\tau, Z), M \in \Gamma^J(L)$$

and $\varphi(\tau, \mathfrak{z})$ has a Fourier expansion:

$$\sum_{\lambda \in L^*, n \geqslant 0} a(n, l) q^n \zeta^\lambda.$$

7 Orbit spaces of Jacobi groups

M. Bertola:

- The orbit space of $J(A_n)$ is isomorphic to covering of Hurwitz space $M_{1,0,n+1}$ (genus 1 with 1 branching point of branching degree n+1) and has a Frobenius manifold structure.

- The orbit space of $J(B_n)$ is isomorphic to covering of Hurwitz space $M^{\mathbb{Z}_2}_{1,0,2n}$ (genus 1 with 1 branching point of branching degree 2n and action $\mathbb{Z}_2$ on torus by involution $v \mapsto -v$) and has a Frobenius manifold structure.

- There is a way to construct the corresponding potentials and flat coordinates for small n. For $J(G_2)$ it is also possible and follows from the case $J(A_2)$.

I. Satake: potential and flat coordinates for $J(E_6)$ (but without isomorphism with any Hurwitz space).

Question: is it possible to obtain similar results for C_n, D_n and F_4?

Thank you for your attention!