A class of solutions of Witten–Dijkgraaf–Verlinde–Verlinde equations

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Overview

Let $\mathcal{A}$ be a finite collection of covectors $\alpha \in V^*$, $V \cong \mathbb{C}^n$. We are interested in solutions

$$F(x_1, \dots, x_n) = \sum_{\alpha \in \mathcal{A}} \alpha(x)^2 \log \alpha(x), \quad x = (x_1, \dots, x_n) \in V,$$

of generalized Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of the form

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i, \quad 1 \leq i, j, k \leq n,$$

where F_i is the $n \times n$ matrix, whose (pq)-entry is $\frac{\partial^3 F(x)}{\partial x_i \partial x_p \partial x_q}$.

Origins of these solutions when $\mathcal{A} = \mathcal{R}$ is a Coxeter root system:

- **Dubrovin's almost duality** [Dubrovin'03]:
  F is almost dual to polynomial Frobenius manifold structure on the orbit space V/W, where $W = \langle s_\alpha : \alpha \in \mathcal{R} \rangle$ is a finite Coxeter group.
  $u * v = E^{-1} \circ u \circ v$.
- **Seiberg–Witten theory** [Marshakov, Mironov, Morozov'96]:
  F is the perturbative part of Seiberg–Witten prepotential

For general $\mathcal{A}$:

- $\vee$-system [Veselov'98]:
  Geometric reformulation of the property that F is a soltion.

(Rational) $\vee$-system

Consider bilinear form on V given by

$$G_{\mathcal{A}}(u, v) = \sum_{\alpha \in \mathcal{A}} \alpha(u)\alpha(v),$$

for any $u, v \in V$, and suppose $G_{\mathcal{A}}$ is non-degenerate.
Let $\varphi : V^* \to V$ be the corresponding isomorphism:

$$G_{\mathcal{A}}(\varphi(\alpha), v) = \alpha(v)$$

for any $\alpha \in V^*$, $v \in V$. Denote $\alpha^\vee = \varphi(\alpha)$.

Definition (Veselov'98)

$\mathcal{A}$ is a (rational) $\vee$-system if for any 2-dimensional plane $\pi \subset V^*$ and any $\alpha \in \mathcal{A} \cap \pi$

$$\sum_{\beta \in \mathcal{A} \cap \pi} \alpha(\beta^\vee)\beta = \lambda\alpha$$

for some $\lambda \in \mathbb{C}$.

Theorem (Veselov'99; F., Veselov'07)

The following conditions are equivalent:

1. $\mathcal{A}$ *is a* $\vee$*-system*
2. F *satisfies WDVV*
3. *Multiplication* $u * v = \sum_{\alpha \in \mathcal{A}} \frac{\alpha(u)\alpha(v)}{\alpha(x)} \alpha^\vee$ *is associative* $(x, u, v \in V; \alpha(x) \neq 0 \ \forall \alpha \in \mathcal{A})$.
4. $G_\pi|_{\pi^\vee \times V} \sim G_{\mathcal{A}}|_{\pi^\vee \times V}$ *if* $|\pi \cap \mathcal{A}| \geq 2$*, where* $G_\pi(u, v) = \sum_{\alpha \in \mathcal{A} \cap \pi} \alpha(u)\alpha(v)$*, and* $G_{\mathcal{A}}(\alpha^\vee, \beta^\vee) = 0$ *if* $\pi \cap \mathcal{A} = \{\alpha, \beta\}$.

Example (Martini, Gragert'99; Veselov'99)

Let $\mathcal{A} = \mathcal{R}$ be a Coxeter root system: $V \cong V^*$, $\forall \alpha, \beta \in \mathcal{R}$ $s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \alpha \in \mathcal{R}$. Then $\langle s_\alpha : \alpha \in \mathcal{R} \rangle$ is a finite Coxeter group. In this case $G_{\mathcal{A}} \sim (\cdot, \cdot)$.

Operations with $\vee$-systems

- **Subsystems.** Let W be a subspace in V^*. Let $\mathcal{B} = \mathcal{A} \cap W$. Then $\mathcal{B} \subset (W^\vee)^*$.

Theorem (F., Veselov'07)

Let $\mathcal{A}$ be a $\vee$-system. Then subsystem $\mathcal{B}$ is a $\vee$-system provided that $G_\mathcal{B}|_{W^\vee}$ is non-degenerate.

- **Restrictions.** Let $\mathcal{B}$ be a subsystem in $\mathcal{A}$. Define $W_\mathcal{B} = \{x \in V \colon \beta(x) = 0 \ \forall \beta \in \mathcal{B}\} \subset V$. Consider resitriction $\pi_\mathcal{B}(\mathcal{A}) = \{\alpha|_{W_\mathcal{B}} \colon \alpha \in \mathcal{A}, \alpha|_{W_\mathcal{B}} \neq 0\} \subset {W_\mathcal{B}}^*$.

Theorem (F., Veselov'05)

Let $\mathcal{A}$ be a $\vee$-system. Then restriction $\pi_\mathcal{B}(\mathcal{A})$ is also a $\vee$-system provided that $G_\mathcal{A}|_{W_\mathcal{B}}$ is non-degenerate.

Example (Chalykh, Veselov'01)

$\mathcal{A} = \{c_i c_j (e^i - e^j) \colon 1 \leq i < j \leq n\}$ is a $\vee$-system for generic c_i.

Trigonometric solutions of WDVV equations

Let $\mathcal{A} \subset V^*$. Let $c : \mathcal{A} \to \mathbb{C}$, denote $c_\alpha = c(\alpha)$. Let

$$F^t(x_1, \ldots, x_n, y) = \mu \sum_{\alpha \in \mathcal{A}} c_\alpha f(\alpha(x)) + \frac{1}{3} y^3 + y \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(x)^2,$$

where $f(z) = \frac{1}{6} i z^3 + \frac{1}{4} Li_3(e^{-2iz})$, $\frac{d^3 Li_3(z)}{dz^3} = \cot z$; $\mu \in \mathbb{C}$.
Origins of these solutions:

- **Dubrovin's almost duality**
  F^t is (expected to be) almost dual to Frobenius manifold structure on the (extended) affine Weyl group orbit space $\mathcal{M} = (V \oplus \mathbb{C})/\widehat{W}$ ($\mathcal{M}$ is given by Dubrovin, Zhang, Zuo, Strachan [1998-2015]; F^t partly confirmed in [Riley, Strachan'07])
- **Seiberg–Witten theory** [Marshakov, Mironov, Morozov'96]: F^t is the perturbative part of 5d Seiberg–Witten prepotential
- **Reductions of hydrodynamic chains**
  Similar solutions [Pavlov'06]

Quantum cohomology of ADE resolutions

- **Quantum Cohomology** of resolutions of ADE singularities

Let Y be the minimal resolution of ADE singularity $\mathbb{C}^2/\Gamma$, where $\Gamma \subset SL(2,\mathbb{C})$ is of ADE type (via McKay correspondence). Let $V = H^2(Y,\mathbb{Z})$.
Diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^2$ lifts to $\mathbb{C}^*$ action on Y. Then equivariant quantum cohomology $QH^*(Y,\mathbb{Z})$ are generated by 1 and exceptional divisors $E_1, \ldots, E_n$ over $\mathbb{C}[y][[x_1, \ldots, x_n]]$, where x_i are quantum parameters corresponding to E_i.

Theorem (Bryan, Gholampour'07)

Quantum product in $QH_{\mathbb{C}^*}(Y,\mathbb{Z})$ is governed by genus 0 Gromov–Witten potential F^t.

Trigonometric $\vee$-systems

Suppose $\mathcal{A}$ belongs to a lattice.
For each $\alpha \in \mathcal{A}$ decompose non-collinear vectors from $\mathcal{A}$,

$$\mathcal{A} = \sqcup_s \Gamma_\alpha^s,$$

where each "series" Γ_α^s satisfies the following property:
for any $\beta, \gamma \in \Gamma^s$ $\beta + \epsilon\gamma = m\alpha$ for some $\epsilon \in \{1, -1\}$, $m \in \mathbb{Z}$.

Example

Let $\mathcal{A} = BC_2^+ = \{e^1, 2e^1, e^2, 2e^2, e^1 + e^2, e^1 - e^2\}$. Then

$$\Gamma_{2e^1}^1 = \{e^1 + e^2, e^1 - e^2\}, \Gamma_{2e^1}^2 = \{e^2\}, \Gamma_{2e^1}^3 = \{2e^2\};$$

$$\Gamma_{e^1}^1 = \{e^1 + e^2, e^2, e^1 - e^2\}, \Gamma_{e^1}^2 = \{2e^2\};$$

$$\Gamma_{e^1-e^2}^1 = \{e^1, e^2\}, \Gamma_{e^1-e^2}^2 = \{2e^1, 2e^2\}, \Gamma_{e^1-e^2}^3 = \{e^1 + e^2\};$$

etc.

Define $G_{\mathcal{A}}(u,v) = G_{(\mathcal{A},c)}(u,v) = \sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(u)\alpha(v), \quad u,v\in V.$
Suppose $G_{\mathcal{A}}$ is non-degenerate so we have $G_{\mathcal{A}}\colon V \xrightarrow{\sim} V^*$. Let $\alpha^\vee = G_{\mathcal{A}}^{-1}(\alpha)$.

Definition (F.'08)

$(\mathcal{A}, c)$ is a trigonometric $\vee$-system if for any $\alpha\in\mathcal{A}$ and for any α-series Γ_α^s

$$\sum_{\beta\in\Gamma_\alpha^s} c_\beta \beta(\alpha^\vee)\beta = \lambda\alpha$$

for some $\lambda\in\mathbb{C}$.

Proposition (F.'08)

Let $(\mathcal{A}, c)$ be a trigonometric $\vee$-system. Then $\mathcal{A}^r = \{\sqrt{c_\alpha}\alpha\colon \alpha\in\mathcal{A}\}$ is a rational $\vee$-system.

WDVV equations:

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i, \quad 1 \le i, j, k \le n+1,$$

where F_i is the $(n+1) \times (n+1)$ matrix, whose (pq)-entry is $\frac{\partial^3 F(x)}{\partial x_i \partial x_p \partial x_q}$ with $x_{n+1} = y$. $F_{n+1} = constant = 2diag(G_{\mathcal{A}}, 1)$

Theorem (F.'08)

Suppose $(\mathcal{A}, c)$ is a trigonometric $\vee$-system. Suppose there exists $\mu \in \mathbb{C}$ such that

$$\sum_{\alpha,\beta \in \mathcal{A}_+} \left(c_\alpha c_\beta \left(\frac{1}{4}\mu^2 \alpha(\beta^\vee) - 1 \right) \right) (\alpha \wedge \beta)(\alpha \wedge \beta) = 0. \quad (1)$$

Then F^t satisfies WDVV.
Conversely, suppose that F^t satisfies WDVV. Then
1) Relation (1) holds.
2) Let $\alpha \in \mathcal{A}$ be such that $\mathcal{A} \cap \langle \alpha \rangle \subseteq \{\pm\alpha\}$. Then trigonometric V-system conditions for the series Γ_α^s are satisfied.

Operations with trigonometric $\vee$-systems/solutions

- **Subsystems.** Let W be a subspace in V^*. Let $\mathcal{B} = \mathcal{A} \cap W$. Then $\mathcal{B} \subset (W^\vee)^*$.

Theorem (Alkadhem, F.)

Let $\mathcal{A}$ be a trigonometric $\vee$-system. Then subsystem $\mathcal{B}$ is a trigonometric $\vee$-system provided that $G_\mathcal{B}|_{W^\vee}$ is non-degenerate.

- **Restrictions.** Let $\mathcal{B}$ be a subsystem in $\mathcal{A}$. Define $W_\mathcal{B} = \{x \in V : \beta(x) = 0 \ \forall \beta \in \mathcal{B}\} \subset V$. Consider resitriction $\pi_\mathcal{B}(\mathcal{A}) = \{\alpha|_{W_\mathcal{B}} : \alpha \in \mathcal{A}, \alpha|_{W_\mathcal{B}} \neq 0\} \subset {W_\mathcal{B}}^*$.

Theorem (Alkadhem, F.)

Let $(\mathcal{A}, c)$ define a solution of WDVV. Then restriction $(\pi_\mathcal{B}(\mathcal{A}), c)$ also defines a trigonometric solution of WDVV provided that $G_\mathcal{A}|_{W_\mathcal{B}}$ is non-degenerate.

Examples

A Goal: to classify rational/trigonometric $\vee$-systems.
Trigonometric $\vee$-systems are non-trivial in $\mathbb{C}^2$ but there are fewer solutions in higher dimensions.

Example (Martini, Hoevenaars'03; Alkadhem, F.)

Let $\mathcal{R}$ be a crystallographic root system with Weyl group W. Let $c : \mathcal{R} \to \mathbb{C}$ be W-invariant. This is a trigonometric $\vee$-system which also defines a solution of WDVV for suitable μ.

Example

$\mathcal{A} = \{e^i - e^j : 1 \leq i < j \leq n\}$, $c(e^i - e^j) = c_i c_j$ is a trigonometric $\vee$-system for generic $c_i \in \mathbb{C}$. It defines a solution of WDVV for a suitable value of μ.

Example (Alkadhem, F.)

Let $V \cong \mathbb{C}^4$. Let

$$\mathcal{A} = \{e^i : 1 \leq i \leq 4\} \cup \{e^i \pm e^j : 1 \leq i < j \leq 3\} \cup \{\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4)\}.$$

Let

$$c(e^i \pm e^j) = r, \quad c(\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4)) = s,$$

$$c(e^1, e^2, e^3) = 2r + s, \quad c(e^4) = \frac{s(s-2r)}{4r+s}.$$

Then $(\mathcal{A}, c)$ is a trigonometric $\vee$-system for generic $r, s \in \mathbb{C}$. It defines solution F^t with $\mu = 6\sqrt{3}(2r+s)(4r+s)^{-1/2}$.

Elliptic solutions

Let

$$f^e(z,\tau) = \frac{1}{(2\pi i)^3}(\mathcal{L}i_3(e^{2\pi i z}, q) - \mathcal{L}i_3(1,q)),$$

$q = e^{2\pi i \tau}$. Then

$$\frac{\partial^3 f^e}{\partial z^3} = -\frac{1}{2\pi i}\frac{\partial \log \theta_1(z,\tau)}{\partial z},$$

where

$$\theta_1(z,\tau) = -i(e^{\pi i z} - e^{-i\pi z})q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2\pi i z})(1-q^n e^{-2\pi i z});$$

$$(\frac{1}{2\pi i})^3 \frac{d^3}{d\tau^3}\mathcal{L}i_3(1,q) = \frac{1}{120}E_4(\tau).$$

Let

$$F^e = \frac{1}{2}\tau u^2 - \frac{1}{2}u\sum_{i=0}^{n} x_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} f^e(x_i - x_j) - (n+1)\sum_{j=0}^{n} f^e(x_i),$$

where $x_0 = -\sum_{i=1}^{n} x_i$. Let $x_{n+1} = \tau, x_{n+2} = u$.

Theorem (Riley, Strachan'06)

Function F^e satisfies WDVV equations

$$F_i^e(F_j^e)^{-1}F_k^e = F_k^e(F_j^e)^{-1}F_i^e, \quad (1 \leq i, j, k \leq n+2).$$

Furtermore, F^e is almost dual to Bertola's Frobenius manifold on the space of orbits of Jacobi group on Jacobi forms on $V \oplus \mathbb{C} \oplus \mathbb{H}$.

It is expected that any trigonometric ∨-system $(\mathcal{A}, c)$ gives rise to elliptic solution of WDVV in $V \oplus \mathbb{C}^2$ similar to F^e provided that

$$\sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(u)^4 \sim G_{\mathcal{A}}(u,u)^2.$$