A computer algorithm for the BGG resolution

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Table of Contents

Sheaf cohomology I

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In particular, for any character $\lambda : T \to \mathbb{C}^*$ we get a line bundle $\mathscr{L}_\lambda$ over G/B.

If $\mathcal{E}$ is a line bundle, the cohomology is given by the famous Borel-Weil-Bott theorem (in characteristic zero). In order to state it, we need to introduce some notations.

Sheaf cohomology II

If P is the weight lattice, $\lambda \in P$ and $w \in W$, let $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $2\rho = \sum_{\alpha \in \Phi^+} \alpha$. We say that λ is *dot-regular* if the stabilizer is trivial for the dot-action, and dot-singular else.

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This problem also is related to Hochschild cohomology of the small quantum group, which was the initial motivation. We will give more details later (if time permits).

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This boils down to finding a $\mathfrak{h}$-graded projective resolution of the $U(\mathfrak{n})$-module $E^* \otimes L(\lambda)$. We can use the *BGG resolution* to compute it.

The BGG resolution

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ a Cartan subalgebra contained in a Borel subalgebra. Let $\lambda \in \mathfrak{h}^*$.

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Consider the one-dimensional $U(\mathfrak{b})$ module $\mathbb{C}_\lambda$ where $\mathfrak{h}$ acts by λ and $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ by zero. The *Verma module* associated to λ is the module $M(\lambda) := Ind_{U(\mathfrak{b})}^{U(\mathfrak{g})} \mathbb{C}_\lambda$.

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Theorem (Bernstein-Gelfand-Gelfand, Rocha-Caridi)

If $\lambda \in P^+$ there is an exact sequence

$$0 \to M(w_0 \cdot \lambda) \to \cdots \to \bigoplus_{\ell(w)=k} M(w \cdot \lambda) \to \cdots \to M(\lambda) \to L(\lambda) \to 0$$

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Given a B-module E and $\lambda \in P^+$, we now present the complex $BGG^\bullet(\lambda, E)$ which computes $Hom_G(L(\lambda), H^\bullet(X, \mathcal{E}))$.

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In particular, $BGG^\bullet(\lambda) \otimes E^*$ is a projective resolution of $L(\lambda) \otimes E^*$. The cohomology we wanted to compute is

$$Ext^\bullet_{\mathfrak{n}}(\mathbb{C}, E \otimes L(\lambda)^*)^{\mathfrak{h}} \cong Ext^\bullet_{\mathfrak{n}}(E^* \otimes L(\lambda), \mathbb{C})^{\mathfrak{h}}$$

hence we can use the BGG resolution.

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which is simply the μ-weight space of E. Now recall that in the BGG resolution, the weights that appears are on the form $w \cdot \lambda$ for $\ell(w) = k$.

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$$BGG^k(\lambda, E) = \bigoplus_{\ell(w)=k} E[w \cdot \lambda]$$

Maps between Verma modules

We quickly describe the maps between Verma modules.

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In the BGG complex, the maps are hence scalars between $M(w \cdot \lambda) \to M(w' \cdot \lambda)$ for $\ell(w') = \ell(w) - 1$. It turns out that the scalars are nonzero iff $w' \leq w$ (in the Bruhat order). Moreover, the scalars can be picked in ± 1.

Maps in the BGG complex

Hence, the condition $d^2 = 0$ becomes the following condition : for each "square" in the Bruhat graph

$$\begin{tikzcd} & w' \arrow[dr, "\varepsilon_2"] & \\ w \arrow[ur, "\varepsilon_1"] \arrow[dr, "\varepsilon_3"'] & & w''' \\ & w'' \arrow[ur, "\varepsilon_4"'] & \end{tikzcd}$$

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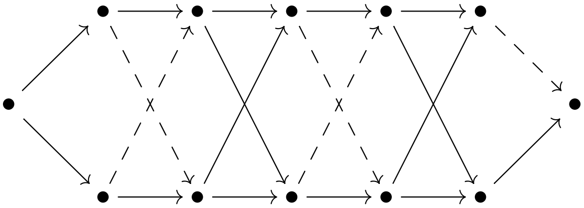
Proposition

A choice of signs exists, and different choices of signs give isomorphic BGG complexes.

Now, we can just define the differential $E[\mu] \to E[\mu']$ as the action by $v \mapsto \pm f_{\mu,\mu'} \cdot v$ (with appropriate choice of signs).

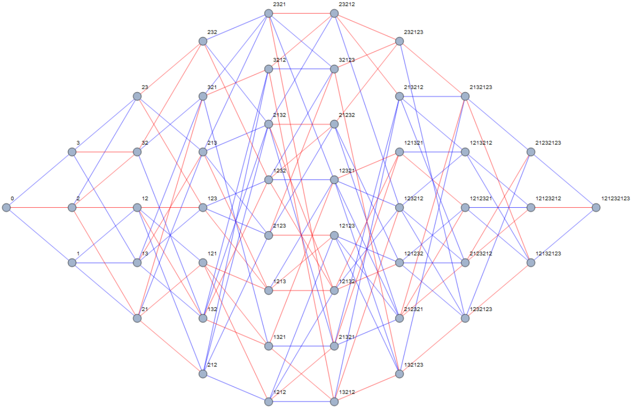
Maps in the BGG complex II

Here is an example of signs choice for $\mathfrak{g} = \mathfrak{g}_2$:



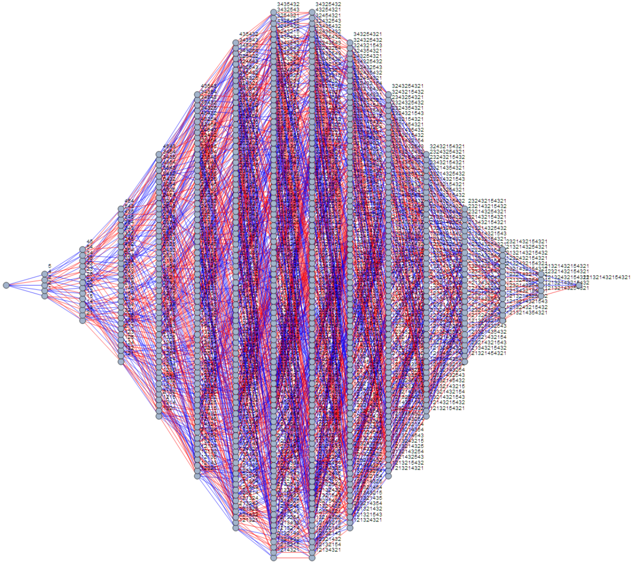
Signs in the Bruhat graph

Here is another example for type A_3 :



Signs in the Bruhat graph

Here is a more complicated example for type A_5 :



Description of the algorithm

We now present the main steps of the algorithm, given a B-module E.

- Compute all $\lambda \in P^+$ so that $L(\lambda)$ appears in $H^\bullet(X, \mathcal{E})$ (i.e compute the dot-orbit of the set of weights of E).

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For example let us consider the $\mathfrak{b}$-module $E = \mathfrak{n}$ for $\mathfrak{g} = \mathfrak{sl}_3$. We have $\mathcal{E} = \Omega^1_X$ and hence we know that $H^1(X, \mathcal{E}) \cong \mathbb{C}^2$.

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It is also easy to check that we only need to consider $\lambda = 0$.

The corresponding BGG complex is given by
$BGG^0(0, \mathfrak{n}) = \mathfrak{n}[0] = 0$,
$BGG^1(0, \mathfrak{n}) = \mathfrak{n}[s_1 \cdot 0] \oplus \mathfrak{n}[s_2 \cdot 0] = \mathfrak{n}[-\alpha_1] \oplus \mathfrak{n}[-\alpha_2]$ and
$BGG^2(0, \mathfrak{n}) = 0$, i.e

$$0 \to \mathbb{C} \oplus \mathbb{C} \to 0$$

Applications

We implemented the algorithm on a computer, and obtained several results, up to rank 5. We mention an application to Hochschild cohomology of flag varieties.

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Theorem (Kontsevitch)

Twisting the HKR isomorphism by the Todd class induces an algebra isomorphism $HH^\bullet(X) \cong HT^\bullet(X)$.

Hochschild cohomology of flag variety

Theorem (H., Vorhaar)

If G has rank 3, $X = G/B$ is "Hochschild affine", i.e $H^i(X, \wedge^j T_X) = 0$ for $i > 0$. For rank 4 and each type, there is a parabolic P such that G/P is not Hochschild affine.

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Corollary

For rank 3, the twist by the Todd class is trivial, hence we have an isomorphism of Gerstenhaber algebras $HH^\bullet(X) \cong HT^\bullet(X)$ given by the HKR map.

Certain class of flag varieties (for example Grassmannians) are Hochschild affine. We hope that we can find a more explicit description of $HH^\bullet(X)$ in future work.

Applications to small quantum group

We also mention an application to the small quantum group $\mathfrak{u}_q(\mathfrak{g})$. This is a finite-dimensional Hopf algebra introduced by Lusztig, which is a quantum analogous of the first Frobenius kernel in modular representation theory.

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where $\widetilde{\mathcal{N}} = T^*(G/B)$ is the Springer resolution, and k is a certain grading induced by a $\mathbb{C}^*$-action and $\mathfrak{u}_0$ is the *principal block* of $\mathfrak{u}_q(\mathfrak{g})$. One can use our algorithm to compute the right-hand side.

The center of the small quantum group

Notice in particular that $HH^0(\mathfrak{u}_0)$ is simply the center of $\mathfrak{u}_0$. It is a very interesting object, possibly connected to many other areas of mathematics than representation theory. We mention a conjecture by Lachowska-You :

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There is an isomorphism of bigraded W-modules :

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This is true for $\mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{sl}_4$ and $\mathfrak{b}_2$. We obtained the $\mathfrak{g}_2$ case as well.

Theorem (H., Vorhaar)

The conjecture holds for $\mathfrak{g} = \mathfrak{g}_2$.

Concluding remarks

First, let us mention that the maps in the BGG complex themselves are of interest, since they represent differential operators between homogenous line bundles on G/B. We hope to be able to relate our formula with existing work.

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In particular, a generalisation of our algorithm to generalized Verma modules and parabolic category $\mathcal{O}$ could give explicit formulas for differential operators between homogeneous vector bundles on any G/P that might be of interest.

Thank you for your attention !