CHALMERS UNIVERSITY OF TECHNOLOGY | UNIVERSITY OF GOTHENBURG

XYZ correlations and Painlevé VI

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Goal: Compute certain correlations of XYZ spin chain exactly for finite systems.

“Compute” means express in terms of algebraic solutions to Painlevé VI.

In statistical mechanics, exact results for finite size systems are very rare, and mostly available for free-fermionic systems.

In our case, they probably exist because of supersymmetry.

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XYZ spin chain

Chain with L particles. Hilbert space $(\mathbb{C}^2)^{\otimes L}$.

Hamiltonian

$$H^{\mathsf{XYZ}} = -\frac{1}{2}\sum_{j=1}^{L}\Big(J_x\,\sigma_j^x\sigma_{j+1}^x + J_y\,\sigma_j^y\sigma_{j+1}^y + J_z\,\sigma_j^z\sigma_{j+1}^z\Big)\,.$$

J_x, J_y, J_z real parameters.

σ_j^x Pauli matrix acting on j-th tensor factor.

$\sigma_{L+1}^x = \sigma_1^x$ periodic boundary conditions.

Supersymmetric/Combinatorial/Stroganov case

$$J_x J_y + J_x J_z + J_y J_z = 0$$

In this special case, Baxter (1972) found that ground state energy (lowest eigenvalue of H^{XYZ})

$$E_0 \sim -\frac{L}{2}(J_x + J_y + J_z), \qquad L \to \infty.$$

Stroganov ("The importance of being odd", 2001) observed that if L is odd then

$$E_0 = -\frac{L}{2}(J_x + J_y + J_z).$$

Proved by Hagendorf and Liénardy (2018) using supersymmetry.

$$H^{\text{XYZ}} = E_0 + QQ^\dagger + Q^\dagger Q$$

(on subspace of $V^{\otimes L}$) where $Q : V^{\otimes L} \to V^{\otimes (L+1)}$.

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Combinatorics

For the supersymmetric XXZ chain ($J_x = J_y = 1$, $J_z = -1/2$) there are deep connections to enumeration of alternating sign matrices and plane partitions (Razumov–Stroganov…).

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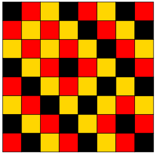

Correlation functions

We will assume

- Periodic boundary
- $J_x J_y + J_x J_z + J_y J_z = 0$ (SUSY case)
- $L = 2n + 1$ odd

$|\Psi\rangle$ ground state of H^{XYZ} with even number of up spins.
We compute nearest neighbour correlations

$$C^x = \frac{\langle \Psi | \sigma_j^x \sigma_{j+1}^x | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad C^y = \cdots, \quad C^z = \cdots.$$

Independent of j.
We give our result near the end of the talk,
but first we survey some earlier work.

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Transfer matrix and Q-operator

Baxter's parametrized (J_x, J_y, J_z) by elliptic functions depending on (η, τ). The SUSY case is $\eta = \pi/3$.

The transfer matrices $\mathbb{T}(u)$ of the 8-vertex model give a one-parameter family of operators (depending also on (η, τ)) commuting with H^{XYZ}.

Baxter also introduced Q-operators $\mathbf{Q}(u)$, which commute with H^{XYZ} and satisfy

$$\mathbf{T}(u)\mathbf{Q}(u) = \theta_1(u-\eta|\tau)^L \mathbf{Q}(u+2\eta) + \theta_1(u+\eta|\tau)^L \mathbf{Q}(u-2\eta).$$

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Eigenvalue of Q-operator

Bazhanov and Mangazeev (2005, 2006) studied the ground state eigenvalue $Q(u)$ of the Q-operator $\mathbf{Q}(u)$.

Under the same conditions ($L = 2n+1$ odd, periodic boundary, $J_xJ_y + J_xJ_z + J_yJ_z = 0$) they found intriguing connections to two systems:

- Painlevé VI equation (PVI)
- Non-stationary Lamé or Quantum Painlevé VI equation (QPVI)

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Painlevé VI

PVI is the 4-parameter family of nonlinear ODEs:

$$\frac{d^2x}{ds^2} = \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-s}\right)\left(\frac{dx}{ds}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{x-s}\right)\frac{dx}{ds}$$
$$+\frac{x(x-1)(x-s)}{s^2(s-1)^2}\left(\alpha_1-\alpha_2\frac{s}{x^2}+\alpha_3\frac{s-1}{(x-1)^2}+\left(\alpha_4-\frac{1}{2}\right)\frac{s(s-1)}{(x-s)^2}\right).$$

This can be brought to a simpler, ellliptic, form.

Define $\tau=\tau(s)$ so that

$$\mathbb{C}[x,y]/(y^2-x(x-1)(x-s)) \simeq \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$$

and let

$$x=\frac{\wp(q|\tau)-e_1}{e_2-e_1},$$

(e_j are values of $\wp$ at half-periods).

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Manin's Hamiltonian

Manin (1998) showed that PVI is equivalent to a Hamiltonian system

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}.$$

Here,

$$\tau = 2\pi \mathrm{i} t,$$

$$H = \frac{p^2}{2} - V(q,t),$$

V is Darboux(-Inozemtsev-Treibich-Verdier-$\cdots$) potential

$$V(q,t) = \sum_{j=1}^{4} \alpha_j \wp(q - \gamma_j | 2\pi \mathrm{i} t),$$

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Quantum Painlevé VI

QPVI is the heat/Schrödinger equation with the same potential

$$\psi_t = \frac{1}{2}\psi_{xx} - V\psi,$$

$$V(x,t) = \sum_{j=1}^{4} \alpha_j \wp(x - \gamma_j | 2\pi \mathrm{i} t).$$

Appears in many contexts (Bernard, Etingof–Kirillov, Suleimanov, Fateev–Litvinov–Neveu–Onofri, Nagoya, Langmann–Takemura, Zabrodin–Zotov, Kolb,...).

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Connection to PVI and QPVI

Bazhanov and Mangazeev found that a multiple of $Q(u)$ satisfies QPVI with parameters

$$\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}, 0, 0\right).$$

They found empirically that at special values of u, $Q(u)$ can be expressed in terms of polynomials s_n, $\bar{s}_n$, which satisfy recursions like

$$s_{n+1}s_{n-1} = (\bullet)(s_n s_n'' - (s_n')^2) + (\bullet)s_n s_n' + (\bullet)s_n^2.$$

They identified s_n and $\bar{s}_n$ with tau functions of PVI, corresponding to particular algebraic solutions of PVI with parameters

$$\left(\frac{n^2}{2}, \frac{n^2}{2}, 0, 0\right) \qquad \text{for } s_n, \qquad \left(\frac{n^2}{2}, \frac{n^2}{2}, \frac{1}{2}, \frac{1}{2}\right) \qquad \text{for } \bar{s}_n.$$

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Proofs

Rigorous proofs of these claims of Bazhanov and Mangazeev were given in R. 2015, with a generalization to general parameters $\alpha_j = k_j(k_j + 1)/2$, $k_j \in \mathbb{Z}$, $\sum_j k_j$ even.

Our main result

Parametrize

$$J_x = 1+\zeta, \quad J_y = 1-\zeta, \quad J_z = \frac{\zeta^2-1}{2}.$$

As a function of τ, ζ is Hauptmodul for $\Gamma_0(12)$.

The correlation function C^z is

$$C^z = \frac{\zeta^4-6\zeta^2+13}{(\zeta^2-1)^2} - \frac{\zeta^2}{2(2n+1)^2(\zeta^2-1)^2}\frac{\bar{s}_n(\zeta^{-2})\bar{s}_{-n-1}(\zeta^{-2})}{s_n(\zeta^{-2})s_{-n-1}(\zeta^{-2})}.$$

Almost identical formulas for C^x and C^y.

If $|\zeta| \leq 3$ and $n \to \infty$, then the second term tends to 0 (probably as $\mathcal{O}(n^{-2})$).

Our proof is based on a technical assumption related to the Q-operator, which we have not proved.

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Connection to Painlevé VI

What does $\bar{s}_n\bar{s}_{-n-1}/s_n s_{-n-1}$ mean for Painlevé VI?

It means that we take Okamoto's PVI Hamiltonian (related to Manin's) with parameters

$$\left(\frac{(n+1/2)^2}{2}, \frac{(n+1/2)^2}{2}, 0, 0\right)$$

and plug in a solution to PVI with parameters

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Recall also that the Q-operator eigenvalue satisfies QPVI with parameters

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Our last remark

Manin (1998):

respectively twistor geometry and Frobenius manifolds.

Our last remark concerns some similarity between the (generalized) Lamé potentials in the theory of KdV–type equations and our classically integrable potentials of the non–linear equation (2.2). According to [TV], the former are of the form

$$\sum_{j=0}^{3} \frac{n_j(n_j+1)}{2}\wp(z+\frac{T_j}{2},\tau),$$

whereas according to our discussion the latter have coefficients (proportional to) $(n_j^2)/2$ or $(n_j+\frac{1}{2})^2/2$. Is there a direct connection between the two phenomena?

References

[D] B. Dubrovin. *Geometry of 2D topological field theories.* In: Springer LNM

Although all three cases appear in our study, conceptual understanding of the relation is still lacking.