Problem session:
Integrable systems and modular forms

Integrable systems and automorphic forms

Sirius Mathematics Center, Sochi

25 February 2020

Plan:

- Differential equations for modular forms.
- Degenerations of modular forms.
- Modular forms and the Odesskii-Sokolov construction.
- Quantisation of modular forms.

Differential equations for modular forms

Example: The Chazy equation

$$f''' + 2ff'' - 3(f')^2 = 0$$

possesses the symmetry group $SL(2,\mathbb{R})$ which acts with an open orbit on the solution space. The generic representative from the open orbit is $f(t) = e_2(it/\pi)$ where $e_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi i n\tau}$ is the Eisenstein series. This solution is invariant under a discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$.

Problem: Given an automorphic form f on a discrete subgroup $\Gamma \subset G$ of a Lie group G, construct an involutive G-invariant system of PDEs S_f such that the action of G on the solution space of S_f possesses an open orbit generated by f. In particular, construct such systems for even theta constants.

Degenerations of modular forms

Example: The Chazy equation

$$f''' + 2ff'' - 3(f')^2 = 0$$

possesses the generic solution $f(t) = e_2(it/\pi)$ (with no continuous stabiliser in $SL(2, \mathbb{R})$), as well as degenerate solutions $f = 1$ and $f = 0$ (stabilisers of dimension one and two, respectively).

Problem: Let f be an automorphic form on a discrete subgroup $\Gamma \subset G$, and let S_f be the corresponding involutive G-invariant system of PDEs for f. Classify degenerate orbits of the action of the symmetry group G on the solution space of the system S_f. Solutions corresponding to non-generic orbits can be viewed as degenerations of f, and are most important from the point of view of their potential applications. In particular, classify non-generic integrable Hirota type equations and non-generic integrable Lagrangians.

Modular forms and the Odesskii-Sokolov construction

The Odesskii-Sokolov construction parametrises broad classes of dispersionless integrable systems by Appell's hypergeometric functions. For special values of parameters (where the monodromy group is a lattice) the corresponding integrable systems can be expressed via modular forms. These special cases include particularly interesting examples such as integrable Euler-Lagrange equations, Hirota type equations, Hamiltonian systems, Godunov systems and so on.

Problem: Classify dispersionless integrable systems corresponding to modular cases of the Odesskii-Sokolov construction (based on Mostov-Deligne). Clarify the 'mathematical physics' behind these modular cases.

Quantisation of modular forms

Example: The dispersionless KP equation $u_{xt} - u_x u_{xx} - u_{yy} = 0$ possesses an integrable dispersive deformation (quantisation)

$$u_{xt} - u_x u_{xx} - u_{yy} - \frac{1}{12} u_{xxxx} = 0,$$

known as the full KP equation.

Problem: Construct integrable dispersive deformations of dispersionless integrable systems that contain modular forms in the coefficients. At the level of dispersionless Lax pairs, dispersive deformations can be viewed as quantisations replacing automorphic symbols by automorphic differential operators. As an example, construct an integrable dispersive deformation of the PDE

$$u_{tt} - \frac{u_{xy}}{u_{xt}} - \frac{1}{6} f(u_{xx}) u_{xt}^2 = 0$$

where f satisfies the Chazy equation.