Higher pullbacks of modular forms on orthogonal groups

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The goal of this talk is to explain how differential operators can be used to compute with orthogonal modular forms, with an emphasis on the relation to Jacobi forms.

I will also talk a little about an application to computing dimensions of spaces of orthogonal modular forms as well as graded ring structures if there is time.

The important ideas are due to Eichler–Zagier on development coefficients of Jacobi forms and Ibukiyama on modular forms attached to groups of higher rank.

Suppose L is a positive-definite, even lattice with quadratic form Q and bilinear form

$$B(x,y) = Q(x+y) - Q(x) - Q(y).$$

A **Jacobi form** of weight k and index L is a holomorphic function

$$\phi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$$

which satisfies

$$\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k e^{2\pi i c Q(z)/(c\tau+d)} \phi(\tau,z), \;\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i \tau Q(\lambda) - 2\pi i B(\lambda,z)} \phi(\tau,z), \;\; \lambda,\mu \in L$$

and whose Fourier series

$$\phi(\tau,z) = \sum_{n\in\mathbb{Z}} \sum_{r\in L'} c(n,r) q^n \zeta^r, \;\; q = e^{2\pi i \tau}, \; \zeta^r = e^{2\pi i \langle r,z\rangle}$$

satisfies $c(n,r) = 0$ if $Q(r) > n$.

Familiar examples: if $L = \{0\}$ then we recover the usual modular forms.

If $L = \mathbb{Z}v$ is a rank one lattice with generator v of norm $Q(v) = m$, then Jacobi forms of lattice index L are the more well-known Jacobi forms of scalar index m.

Eichler–Zagier defined development coefficients: if

$$\phi = \sum_{n,r} c(n,r) q^n \zeta^r$$

is a Jacobi form of index m and weight k, then

$$D_N \phi(\tau) = \sum_{n=0}^{\infty} \sum_{r} G_N^{(k-1)}(r, nm) c(n,r) q^n$$

is a modular form of weight $k+N$. Here G_N^{k-1} is, up to a constant multiple, the Gegenbauer polynomials defined by

$$\sum_{N=0}^{\infty} G_N^s(r,n) t^n = (1 - rt + nt^2)^{-s}.$$

It is a cusp form if $N > 0$.

First examples:

$$D_0\phi(\tau) = \sum_{n,r} c(n,r)q^n;$$

$$D_2\phi(\tau) = \sum_{n,r}(kr^2 - 2nm)c(n,r)q^n;$$

$$D_4\phi(\tau) = \sum_{n,r}\Big((k+1)(k+2)r^4 - 12(k+1)r^2nm + 12n^2m^2\Big)c(n,r)q^n.$$

Note $D_N\phi = 0$ if N is odd.

This can be generalized to arbitrary index. For a linear form $r \in L'$ and a bilinear form B, we define Gegenbauer polynomials $G_N^s(r, B)$ by replacing all products in G_N^s by tensor products and symmetrizing the result. So $G_N^s(r, B)$ is a multilinear N-form on L. More precisely we symmetrize the coefficient of t^N in

$$\sum_{N=0}^{\infty} (1 - rt + Bt^2)^{-s} \in T^*L[|t|]$$

where $T^*L = \mathbb{C} \oplus L' \oplus L' \otimes L' \oplus ...$

Proposition

Let $K \subseteq L$ be any even sublattice and let B be the bilinear form on L. For any $N \in \mathbb{N}_0$, and any $\alpha \in (K^\perp)^{\otimes N}$, if

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{r \in L'} c(n, r) q^n \zeta^r$$

is a Jacobi form of weight k and index L then

$$D_N^K \phi(\tau, z; \alpha) = \sum_{n=0}^{\infty} \Big(\sum_{r \in L'} c(n, r) G_N^{k-1-(\mathrm{rk}\, K)/2} \Big(r|_K, nB|_K/2 \Big)(\alpha) \Big) q^n \zeta^r, \; z \in K \otimes \mathbb{C}$$

is a Jacobi form of weight $k + N$ (and cusp form if $N > 0$) and index K.

Note: this can be derived directly from Eichler–Zagier's result using some tricks like theta decomposition; evaluating along diagonal α; etc.

Lowest index examples:

$$D_0^K \phi(\tau, z) = \sum_{n,r} c(n,r) q^n \zeta^r;$$

$$D_1^K \phi(\tau, z; v_1) = \sum_{n,r} \langle r, v_1 \rangle c(n,r) q^n \zeta^r;$$

$$D_2^K \phi(\tau, z; v_1, v_2) = \sum_{n,r} \Big((k - \frac{1}{2}\mathrm{rk}\, K)\langle r, v_1 \rangle \langle r, v_2 \rangle - n \langle v_1, v_2 \rangle \Big) c(n,r) q^n \zeta^r.$$

Jacobi forms of lattice index are useful to study modular forms for orthogonal groups.

The setup: a signature $(2, \ell)$ even lattice Λ with quadratic form $Q : \Lambda \to \mathbb{Z}$, and suppose Λ splits as

$$\Lambda = L \oplus U,$$

where L has signature $(1, \ell - 1)$ and U is the hyperbolic plane (i.e. $\mathbb{Z}^2$ with quadratic form $(x, y) \mapsto xy$).

(Some things work in greater generality but I take Λ to have this form to keep things easier.)

Let $G = \mathrm{SO}^+(\Lambda \otimes \mathbb{R})$ be the connected component of the identity in $\mathrm{O}(\Lambda \otimes \mathbb{R})$.

The analogue of the upper half-space is the tube over a positive cone. Let $P \subseteq L \otimes \mathbb{R}$ be a connected component of the positive-norm vectors and define

$$\mathbb{H}_L = \{z = x + iy : \ x, y \in L \otimes \mathbb{R}, \ y \in P\}.$$

G acts on $\mathbb{H}_L$ by Möbius transformations. In other words $M \cdot z = w$ if and only if

$$M \cdot \begin{bmatrix} -Q(z) \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} -Q(w) \\ w \\ 1 \end{bmatrix} \text{ in } \mathbb{P}(\Lambda).$$

In this case the *factor of automorphy* $\mathrm{j}(M; z)$ is the constant for which

$$M \cdot \begin{pmatrix} -Q(z) \\ z \\ 1 \end{pmatrix} = \mathrm{j}(M; z) \begin{pmatrix} -Q(w) \\ w \\ 1 \end{pmatrix}.$$

A **modular form** of weight k is a holomorphic function

$$f : \mathbb{H}_L \longrightarrow \mathbb{C}$$

satisfying $f(M \cdot z) = \mathrm{j}(M; z)^k f(z)$ for all M in the **modular group**

$$\Gamma = \Gamma_L = \Big\{ M \in G : \; M \cdot \Lambda = \Lambda, \; M \text{ acts trivially on } \Lambda'/\Lambda \Big\},$$

(in other words the discriminant kernel of Λ) as well as a condition of extending to the cusps (which is automatically satisfied if L is of rank at least 3.)

Remark: we write $f|_k M(z) = \mathrm{j}(M; z)^{-k} f(M \cdot z)$.

A few familiar examples:

(i) If $L = \mathbb{Z}v$ has rank one and the generator v has norm $Q(v) = 1$, then modular forms for Γ_L of weight k are modular forms of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$.

(ii) If $L \cong \mathrm{II}_{1,1}$ is unimodular then modular forms for Γ_L are functions of two variables $f(\tau_1, \tau_2)$, $\tau_1, \tau_2 \in \mathbb{H}$ which are modular forms in each variable and are invariant under $(\tau_1, \tau_2) \mapsto (\tau_2, \tau_1)$.

(iii) If $L \cong \mathrm{II}_{1,1} \oplus A_1(-1)$, for example the lattice of symmetric integral (2×2) matrices with $Q(x) = \det(x)$, then modular forms for Γ_L are essentially the same as Siegel modular forms of degree two for the full group $\mathrm{Sp}_4(\mathbb{Z})$.

Now suppose our lattice L has the form $L = U \oplus L_+(-1)$ where L_+ is positive definite. We can write the upper half-space in the form

$$\mathbb{H}_L = \Big\{ (\tau, z, w) : \ \tau, w \in \mathbb{H}, \ z \in L_+ \otimes \mathbb{C}, \ Q(\mathrm{im}(z)) < \mathrm{im}(\tau) \cdot \mathrm{im}(w) \Big\}.$$

Then orthogonal modular forms for L have Fourier-Jacobi expansions (or Fourier expansions about the one-dimensional cusp corresponding to U):

$$f(\tau, z, w) = \sum_{n=0}^{\infty} \phi_n(\tau, z) s^n, \quad s = e^{2\pi i w}.$$

Each ϕ_n is a Jacobi form of index $L_+(n)$. When $n = 0$ that means ϕ_0 is a modular form in τ which is independent of z.

Now if $K_+ \subseteq L_+$ is a sublattice and $K = U \oplus K^+(-1)$ then we obtain pullback operators on orthogonal modular forms from L to K by taking development coefficients in the Fourier-Jacobi expansion termwise:

$$P_N^K f(\tau, z, w)(\alpha) = \sum_{n=0}^{\infty} D_N^{K_+} \phi_n(\tau, z)(\alpha) s^n, \;\; \alpha \in (K^\perp)^{\otimes N}.$$

This is a natural modification of the Taylor expansion of f about $\mathbb{H}_K$ which produces modular forms. As a special case, note that if f is the *Gritsenko lift* of a Jacobi form ϕ then its pullbacks to K are Gritsenko lifts of development coefficients of ϕ.

Even when we cannot split off a hyperbolic plane from L, it is possible to define pullback maps. (They do not really depend on the lattice; they are equivariant with respect to the real groups $\mathrm{SO}^+(2,\ell)$.) By considering the Fourier expansion we obtain the formula

$$P_N^K F(z) = \sum_{\lambda \in K'} \Big(\sum_{\mu \in (K^\perp)'} c(\lambda,\mu) G_N^{k - \mathrm{rk}\, K/2}(\mu, -Q(\lambda)/2 \cdot B) \Big) \mathbf{q}^\lambda,$$

if

$$F(z) = \sum_\lambda c(\lambda) \mathbf{q}^\lambda, \quad \mathbf{q}^\lambda = e^{2\pi i \lambda(z)}.$$

Here $c(\lambda,\mu) = 0$ if $(\lambda,\mu) \notin L'$.

Some examples.

(i) Let $L = U$ be the standard hyperbolic plane and let K be the span of $(1, 1)$. Then $\mathbb{H} = \mathbb{H}_K \to \mathbb{H}_L = \mathbb{H} \times \mathbb{H}$ is the diagonal inclusion $\tau \mapsto (\tau, \tau)$. The N^{th} pullback of a modular form

$$F = \sum_{i=1}^{n} f_i \otimes g_i \in M_*(\Gamma_U)$$

is (up to a constant) the termwise Rankin-Cohen bracket:

$$P_N F = \sum_{i=1}^{n} [f_i, g_i]_N.$$

In other words the coefficients of the N^{th} Rankin-Cohen bracket of two modular forms of weight k are essentially the same as the Gegenbauer polynomial $G_N^{k-1/2}(x, y)$.

(ii) Let $L = U \oplus A_1(-1)$ and let K be the copy of U. Then

$$\mathbb{H} \times \mathbb{H} = \mathbb{H}_K \to \mathbb{H}_L = \mathbb{H}_2$$

is the embedding along the diagonal $(\tau_1, \tau_2) \mapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$. The N^{th} pullback of a Siegel modular form

$$F\Big(\begin{pmatrix} \tau & z \\ z & w \end{pmatrix} \Big) = \sum_{a,b,c} \alpha(a,b,c) q^a r^b s^c, \; q = e^{2\pi i \tau}, \; r = e^{2\pi i z}, \; s = e^{2\pi i w}$$

is Ibukiyama's operator

$$P_N F(\tau_1, \tau_2) = \sum_{a,b,c} \alpha(a,b,c) G_N^{k-1}(b, ac) q_1^a q_2^c.$$

(iii) Let L be the lattice of symmetric integer (2×2) matrices and let $K = \mathrm{span}_{\mathbb{Z}}(A)$ for some positive definite integer matrix A, so Γ_K contains $\Gamma_0(\det A)$. The embedding

$$\mathbb{H} = \mathbb{H}_K \longrightarrow \mathbb{H}_L = \mathbb{H}_2$$

is the map $\tau \mapsto A\tau$.

(iii), continued. Given integral matrices B_i which are orthogonal to A with respect to

$$\langle X, Y \rangle = \det(X+Y) - \det(X) - \det(Y)$$

and a Siegel modular form $F = \sum_T c(T) q^T$ of weight k we obtain

$$P_0 F(\tau) = F(A\tau) = \sum_T c(T) q^{\mathrm{tr}(TA)},$$

$$P_1 F(\tau; B) = \sum_T c(T) \mathrm{tr}(TB) q^{\mathrm{tr}(TA)},$$

$$P_2 F(\tau; B_1, B_2) = \frac{1}{4\det(A)} \sum_T c(T) \Big((4k+2)\det(A)\mathrm{tr}(TB_1)\mathrm{tr}(TB_2) + \langle B_1, B_2 \rangle \mathrm{tr}(TA)^2 \Big) q^{\mathrm{tr}(TA)}.$$

A numerical example: let Ψ_{10} and Ψ_{35} denote the normalized cusp forms of weight 10 and 35:

$$\Psi_{10} = qs(r - 2 + r^{-1})\Big(1 - 2(q+s)(r + 10 + r^{-1}) \pm ...\Big)$$

$$\Psi_{35} = q^2 s^2 (q-s)(r - r^{-1})\Big(1 - (q+s)(r^2 + 70 + r^{-2}) \pm ...\Big)$$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Both Ψ_{10} and Ψ_{35} vanish on all $A\tau$, $\tau \in \mathbb{H}$. The pullbacks of order one are also zero.

A numerical example, continued. The matrices $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ span $A^\perp$ (which consists exactly of matrices of trace zero). The order two pullbacks of Ψ_{10} and Ψ_{35} to $\{A\tau : \tau \in \mathbb{H}\}$ are the bilinear forms

$$P_2 \Psi_{10} : \begin{pmatrix} (B_1, B_1) \\ (B_1, B_2) \\ (B_2, B_2) \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 21q^2 - 1008q^3 + 22680q^4 \pm ... = 21\Delta^2 \end{pmatrix}$$

$$P_2 \Psi_{35} : \begin{pmatrix} (B_1, B_1) \\ (B_1, B_2) \\ (B_2, B_2) \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 142q^5 - 20448q^6 - 26515944q^7 \pm ... = 142\Delta^5 E_{14} \\ 0 \end{pmatrix}$$

Recall: pullbacks of lifts were lifts of development coefficients. In fact this relation holds for the additive theta lift even when we do not have Fourier-Jacobi expansions. Here the input functions are not Jacobi forms and the development coefficients have to be interpreted somewhat differently.

(Namely the setting of vector-valued modular forms and vector-valued Jacobi forms for Weil representations seems to be more natural.) I will leave out the details here and only give a vague example.

Consider pullbacks of Siegel modular forms of degree two to (Hilbert) modular forms for the group $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{5})})$. This corresponds to the embedding of half-spaces

$$\Phi : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}_2, \quad (\tau_1, \tau_2) \mapsto \Omega^T \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \Omega$$

where $\Omega = \begin{pmatrix} 1 & \omega \\ 1 & \omega' \end{pmatrix}$, $\omega = \frac{1+\sqrt{5}}{2}$, $\omega' = \frac{1-\sqrt{5}}{2}$. This corresponds to the embedding of lattices

$$K = (\mathcal{O}_{\mathbb{Q}(\sqrt{5})}, N) \cong \left(\mathbb{Z}^2, \left(\begin{smallmatrix} 2 & 1 \\ 1 & -2 \end{smallmatrix}\right)\right) \to \left(\mathbb{Z}^3, \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)\right) \cong U \oplus A_1(-1) = L.$$

The Doi-Naganuma lift takes modular forms in $M_k(\rho_K)$ to Hilbert modular forms where $\rho : \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{C}[K'/K]$ is the Weil representation. Here K'/K is the cyclic group of order 5.

I will look at the Siegel Eisenstein series of weight 4 which is the lift of the Jacobi form of index 1

$$E_{4,1} = 1+(\zeta^{-2}+56\zeta^{-1}+126+56\zeta+\zeta^2)q+(126\zeta^{-2}+576\zeta^{-1}+756+576\zeta+126\zeta^2)q^2+...$$

It is more useful to rewrite this as a vector-valued Jacobi form of index 1/5 transforming under ρ_K and a permutation representation of the Heisenberg group:

$$\begin{aligned} E_{4,1/5} = &\Big(1 + 126q + (56\zeta^{-1} + 756 + 56\zeta)q^2 + ...\Big)\mathfrak{e}_0 \\ &+ \Big(q^{1/5}\zeta^{2/5} + q^{6/5}(56\zeta^{-3/5} + 126\zeta^{2/5}) + ...\Big)\mathfrak{e}_1 \\ &+ \Big(q^{4/5}(56\zeta^{-1/5} + \zeta^{4/5}) + q^{9/5}(\zeta^{-6/5} + 576\zeta^{-1/5} + 126\zeta^{4/5}) + ...\Big)\mathfrak{e}_2 \\ &+ \Big(q^{4/5}(\zeta^{-4/5} + 56\zeta^{1/5}) + q^{9/5}(126\zeta^{-4/5} + 576\zeta^{1/5} + \zeta^{6/5}) + ...\Big)\mathfrak{e}_3 \\ &+ \Big(q^{1/5}\zeta^{-2/5} + q^{6/5}(126\zeta^{-2/5} + 56\zeta^{3/5}) + ...\Big)\mathfrak{e}_4 \end{aligned}$$

Now the pullbacks of Siegel's E_4 are lifts of development coefficients of $E_{4,1/5}$. For example the Hilbert Eisenstein series of weight 4 is the lift of $E_{4,1/5}(\tau, 0)$ and Gundlach's product of ten theta-constants is the lift of $\frac{d}{dz}\Big|_{z=0} E_{4,1/5}(\tau, z)$.

I want to finish by discussing modular forms for the lattice $L = U \oplus \mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{-7})$ and $\mathcal{O}_K$ carries the negative norm-form. Lattices of this type are interesting because they correspond to (symmetric) Hermitian modular forms (i.e. for the group $\mathrm{SU}_{2,2}(\mathcal{O}_K)$). I found the pullback operators useful to compute with these modular forms. This is an extension of an argument used by Dern and Krieg to approach such problems.

There is a special modular form (a Borcherds product) b_7 of weight 7 whose divisor consists of a third-order zero on the Heegner divisor $\mathcal{H}_1$ of discriminant 1, and a simple zero on the Heegner divisor $\mathcal{H}_2$ of discriminant 2. The pullbacks to those can be understood as (symmetric) paramodular forms for the groups $K(1) = \mathrm{Sp}_4(\mathbb{Z})$ and $K(2)$, respectively.

One can show that the odd-order pullbacks to $\mathcal{H}_1$ are zero, and also that if f is a modular form which vanishes to order at least N on $\mathcal{H}_2$ then its pullbacks to $\mathcal{H}_1$ of all orders vanish to order $2N$ along the diagonal and are therefore multiples of Ψ_{10}^N (where Ψ_{10} is Igusa's cusp form).

Moreover these are essentially the only conditions that the pullbacks of such a Hermitian modular form have to satisfy. (To see this, we use generators for these rings of paramodular forms due to Igusa and Ibukiyama–Onodera, and find preimages).

For even weights we let P_{even} be the tuple of pullbacks

$$P_{even} = (P_0^{\mathcal{H}_1}, P_2^{\mathcal{H}_1}, P_0^{\mathcal{H}_2}, P_1^{\mathcal{H}_2})$$

and for odd weights we define

$$P_{odd} = (P_1^{\mathcal{H}_1}, P_0^{\mathcal{H}_2}).$$

The kernel of P_{even} resp. P_{odd} consists exactly of multiples of b_7.

Then for even weights we obtain the exact sequence

$$0 \longrightarrow \ker(P_{even}) \xrightarrow{\times b_7} M_{2*}(\Gamma_L) \xrightarrow{P_{even}} \mathrm{im}\, P_{even} \longrightarrow 0,$$

and

$$0 \longrightarrow \Psi_{10}^2 M_{2*-20}(K(1)) \oplus \Psi_{10}^2 S_{2*-18}(K(1)) \longrightarrow \mathrm{im}\, P_{even} \longrightarrow M_{2*}^{sym}(K(2)) \oplus M_{2*+1}^{sym}(K(2)) \longrightarrow 0.$$

For odd weights,

$$0 \longrightarrow \ker(P_{odd}) \xrightarrow{\times b_7} M_{2*+1}(\Gamma_L) \xrightarrow{P_{odd}} \mathrm{im}\, P_{odd} \longrightarrow 0$$

and

$$0 \longrightarrow \Psi_{10} M_{2*-9}(K(1)) \longrightarrow \mathrm{im}\, P_{odd} \longrightarrow S_{2*+1}^{sym}(K(2)) \longrightarrow 0.$$

Since Hilbert series are additive in short exact sequences, this reduces the computation of dimensions to modular forms for $\mathrm{Sp}_4(\mathbb{Z})$ and $K(2)$, which are already known.

Altogether the result is

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_L) t^k = \frac{P(t)}{(1-t^6)(1-t^7)(1-t^8)(1-t^{10})(1-t^{12})},$$

where

$$P(t) = 1 + t^4 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{18} + t^{19} + t^{20} + t^{22} + t^{23} + t^{24} + t^{27} - t^{30} - t^{34}.$$

Table: Dimensions

k	1	2	3	4	5	6	7	8	9	10	11	12
$\dim M_k(\Gamma_L)$	0	0	0	1	0	1	1	2	1	3	2	4
$\dim \mathrm{Lifts}_k(\Gamma_L)$	0	0	0	1	0	1	1	2	1	3	2	3

This sort of argument also yields generators for the graded ring of modular forms. Here it shows that certain Maass lifts of weights between 4 and 12 are generators.

Thank you.