Quantized numbers

Sophie Morier-Genoud & Valentin Ovsienko

I Introduction: quantizing sequences

n positive integer

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- Recurrences:

$$[n+1]_q = q\,[n]_q + 1, \qquad\qquad [n+1]_q = [n]_q + q^n$$

Gaussian q-binomial coefficients

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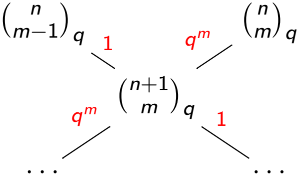

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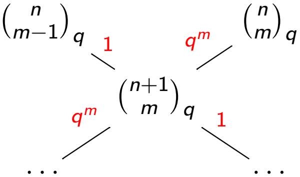

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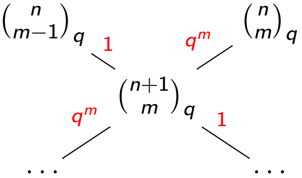

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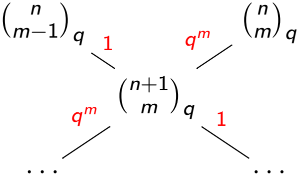

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Conclusion: One does not quantize "6", but *sequences*...

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"Easy q", no good!..

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$$\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)},$$

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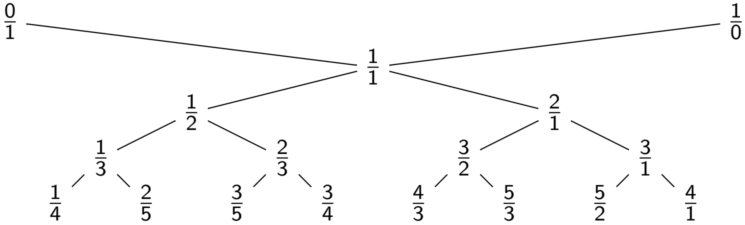
To do that organize $\mathbb{Q}$ as sequence.

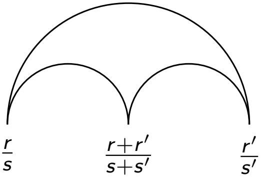
II $\mathbb{Q}$-deformations

Farey, or Stern-Brocot tree



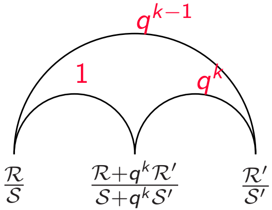
$\dots$

- Elementary triangle:



Definition by recurrence

- **Weighted** triangles of the Farey graph and weighted Farey sum:



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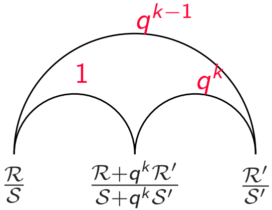

Recurrence: defines q-deformation of every rational $\frac{r}{s} \geq 0$.

Ex: $\left[\frac{1}{2}\right]_q = \frac{q}{1+q}$, $\quad \left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$, $\quad \left[\frac{5}{2}\right]_q = \frac{1+2q+q^2+q^3}{1+q}$.

III Properties of q-rationals

- $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$; $\mathcal{R}$ and $\mathcal{S}$ have **positive integer coefficients**.

Theorem ("Total positivity"): If $\frac{r}{s} > \frac{r'}{s'}$, then the polynomial $\mathcal{R}\mathcal{S}' - \mathcal{S}\mathcal{R}'$ has positive integer coefficients.

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  extends the notion of q-rational to $\frac{r}{s} < 0$.

  Ex: $[-1]_q = -\frac{1}{q}$, $\quad [-2]_q = -\frac{1}{q} - \frac{1}{q^2}$, $\quad \left[-\frac{1}{2}\right]_q = -\frac{1}{q+q^2}$.

Explicit formulas

- $\frac{r}{s}$ is a continued fraction $\frac{r}{s} = [a_1, a_2, a_3, \ldots, a_{2m}]$.

Theorem. $$\left[\frac{r}{s}\right]_q = [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{[a_4]_{q^{-1}} + \cfrac{q^{-a_4}}{\ddots}}}}$$

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- Matrix presentation: *q-deformed generators* of $\mathrm{PSL}(2, \mathbb{Z})$

$$R_q := \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad L_q := \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}, \quad S_q := \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}$$

NB: Quantum Teichmüller theory (Chekhov-Fock,...).

Theorem. $R_q^{a_1} L_q^{a_2} \cdots R_q^{a_{2m-1}} L_q^{a_{2m}} = \begin{pmatrix} q\mathcal{R} & * \\ q\mathcal{S} & * \end{pmatrix}.$

Pb: $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$; $\qquad \mathcal{R} = 1 + R_1 q + \ldots + R_{n-1} q^{n-1} + q^n.$

Combinatorial meaning of coeff. $\mathcal{R}$ (and $\mathcal{S}$) ?

Counting on graphs

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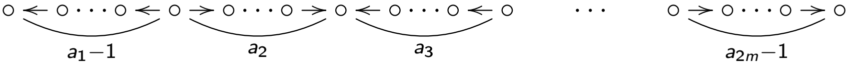

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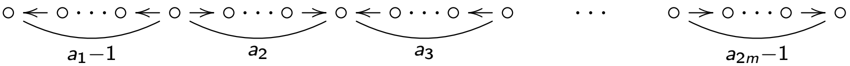

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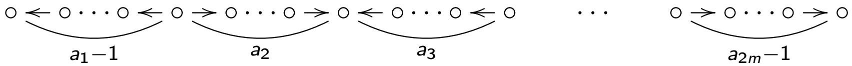

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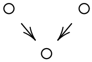
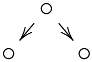
Ex: $\frac{5}{3} = \frac{1+q+2q^2+q^3}{1+q+q^2}$  $\qquad \frac{5}{2} = \frac{1+2q+q^2+q^3}{1+q}$ 

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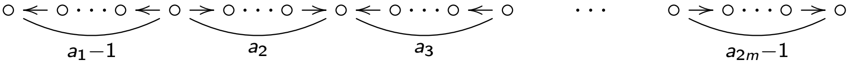

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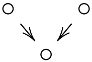
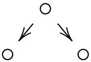
Ex: $\frac{5}{3} = \frac{1+q+2q^2+q^3}{1+q+q^2}$  $\qquad \frac{5}{2} = \frac{1+2q+q^2+q^3}{1+q}$ 

Cor: Coeffs. of $\mathcal{R}$ count subreps. of maximal indecomposable rep.

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Theorem. (i) *Taylor series* of $[x_n]_q$ **stabilize** as n grows.
(ii) Limit Taylor series does not depend on the choice of sequence.

- Property $[x+1]_q = q\,[x]_q + 1$ still holds ($\mathbb{Z}$-action);

  extends $[x]_q$ for $x < 0$, (**Laurent series**).

IV Examples of q-irrationals

The "Golden Ratio"

$$\varphi = \frac{1+\sqrt{5}}{2} = [1, 1, 1, 1, 1, \ldots] \, .$$

is the simplest irrational number. The equation: $\varphi^2 = \varphi + 1$.

The convergents of this continued fraction:

$$\varphi_n = \underbrace{[1, 1, \ldots, 1]}_{n} = \frac{F_{n+1}}{F_n},$$

where F_n is the n^{th} Fibonacci number.

The q-deformations:

$$[\varphi_6]_q = \frac{1 + 2q + 3q^2 + 3q^3 + 3q^4 + q^5}{1 + 2q + 2q^2 + 2q^3 + q^4},$$

$$[\varphi_8]_q = \frac{1 + 3q + 5q^2 + 7q^3 + 7q^4 + 6q^5 + 4q^6 + q^7}{1 + 3q + 4q^2 + 5q^3 + 4q^4 + 3q^5 + q^6},$$

$$[\varphi_9]_q = \frac{1 + 4q + 7q^2 + 10q^3 + 11q^4 + 10q^5 + 7q^6 + 4q^7 + q^8}{1 + 4q + 6q^2 + 7q^3 + 7q^4 + 5q^5 + 3q^6 + q^7}.$$

The coefficients: A123245 of OEIS and its mirror A079487.

The stabilization phenomenon

The Taylor series of the convergents:

$$\begin{aligned}[\varphi_6]_q &= 1 + q^2 - q^3 + 2q^4 - 3q^5 + 3q^6 - 3q^7 + 4q^8 \\ &\quad -5q^9 + 5q^{10} - 5q^{11} + 6q^{12} \cdots \\ [\varphi_8]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 16q^7 + 30q^8 \\ &\quad -55q^9 + 103q^{10} - 195q^{11} + 368q^{12} \cdots \\ [\varphi_9]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 \\ &\quad -82q^9 + 184q^{10} - 414q^{11} + 932q^{12} \cdots\end{aligned}$$

The coefficients stabilize!

q-deformed Golden Ratio

The q-deformation $[\varphi]_q$ is given by the series

$$
\begin{aligned}
[\varphi]_q \;=\;& 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+37q^8\\
&-82q^9+185q^{10}-423q^{11}+978q^{12}-2283q^{13}\\
&+5373q^{14}-12735q^{15}+30372q^{16}-72832q^{17}\\
&+175502q^{18}-424748q^{19}+1032004q^{20}\cdots
\end{aligned}
$$

The coefficients coincide with the sequence A004148 of OEIS called the *generalized Catalan numbers*.

q-deformed equations

The series $[\varphi]_q$ satisfies the equation

$$q\,[\varphi]_q^2 = \left(q^2+q-1\right)[\varphi]_q + 1.$$

This is the q-analogue of $\varphi^2 = \varphi + 1$.

Generating function:

$$\begin{aligned} [\varphi]_q &= \frac{q^2+q-1+\sqrt{q^4+2q^3-q^2+2q+1}}{2q} \\ &= \frac{q^2+q-1+\sqrt{(q^2+3q+1)(q^2-q+1)}}{2q}. \end{aligned}$$

The continued fraction

$[\varphi]_q$ can be written as infinite continued fraction:

$$[\varphi]_q = 1 + \cfrac{q^2}{q + \cfrac{1}{1 + \cfrac{q^2}{q + \cfrac{1}{\ddots}}}} = 1 + \cfrac{1}{q^{-1} + \cfrac{1}{q^2 + \cfrac{1}{q^{-3} + \cfrac{1}{\ddots}}}}$$

NB: The celebrated Rogers-Ramanujan continued fraction

$$R(q) = 1 + \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{\ddots}}}}$$

The "quantum π"

$$
\begin{aligned}
[\pi]_q \quad = \quad & 1 + q + q^2 + q^{10} - q^{12} - q^{13} + q^{15} + q^{16} \\
& -q^{20} - 2q^{21} - q^{22} + 2q^{23} + 4q^{24} + q^{25} \\
& -4q^{27} - 4q^{28} - 2q^{29} + q^{30} + 5q^{31} + 8q^{32} + 3q^{33} \\
& -3q^{34} - 10q^{35} - 12q^{36} - 5q^{37} + 8q^{38} + 19q^{39} + 20q^{40} \\
& +2q^{41} - 18q^{42} - 32q^{43} - 25q^{44} + 31q^{46} + 51q^{47} \\
& +45q^{48} - 7q^{49} - 65q^{50} - 94q^{51} - 57q^{52} + 35q^{53} \cdots
\end{aligned}
$$

The "quantum e"

Euler's number $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots]$

$$
\begin{aligned}
[e]_q \;=\;& 1 + q + q^3 - q^5 + 2q^6 - 3q^7 + 3q^8 - q^9 \\
& -3q^{10} + 9q^{11} - 17q^{12} + 25q^{13} - 29q^{14} + 23q^{15} + 2q^{16} \\
& -54q^{17} + 134q^{18} - 232q^{19} \\
& +320q^{20} - 347q^{21} + 243q^{22} + 71q^{23} \\
& -660q^{24} + 1531q^{25} - 2575q^{26} \\
& +3504q^{27} - 3804q^{28} + 2747q^{29} + 488q^{30} \cdots
\end{aligned}
$$

Observations :

- the coefficients of q^{2+7k} are smaller!
- the signs $+, +$ appear with periodicity 7 !

More examples

Le "silver ratio" $\sqrt{2} = [1, 2, 2, 2, \ldots]$:

$$\left[\sqrt{2}\right]_q = \frac{q^3 - 1 + \sqrt{(q^4 + q^3 + 4q^2 + q + 1)(q^2 - q + 1)}}{2q^2}$$

satisfies $q^2 \left[\sqrt{2}\right]_q^2 - \left(q^3 - 1\right) \left[\sqrt{2}\right]_q = q^2 + 1.$

Square toot of 5:

$$\left[\sqrt{5}\right]_q = \frac{q^5 + q^3 - q^2 - 1 + \sqrt{\begin{array}{l}(q^8 + q^7 + 2q^6 + 3q^5 + 6q^4 \\ +3q^3 + 2q^2 + q + 1)(q^2 - q + 1)\end{array}}}{2q^3},$$

Compare with the golden ratio!

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