Holographic transform : the Rankin-Cohen case

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Multiplicity free branching law

$$\Leftrightarrow$$

$$\dim Hom_{G'}(\pi|_{G'}, \rho_\lambda) = 1$$

In this context, a non zero element $T_\lambda \in Hom_{G'}(\pi|_{G'}, \rho_\lambda)$ is called a *symmetry breaking operator*, and the collection $(T_\lambda)_{\lambda\in\hat{G}'}$ is called a *symmetry breaking transform*.

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One can reverse the arrows in the previous definition, and a non zero element $P_\lambda \in Hom_{G'}(\rho_\lambda, \pi|_{G'})$ is called an *holographic operator*, and the collection $(P_\lambda)_{\lambda \in \hat{G}'}$ is called an *holographic transform*.

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$\rightarrow$ Spectral analysis.

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$\rightarrow$ Spectral synthesis.

Let $\lambda \in \mathbb{N}\backslash\{0;1\}$, and define the weighted Bergmann space on the Poincaré upper half-plane :

$$H^2_\lambda(\Pi) = \mathcal{O}(\Pi) \cap L^2(\Pi, y^{\lambda-1}\ dxdy)$$

We then define the holomorphic discrete series representation π_λ of $SL_2(\mathbb{R})$ for $f \in H^2_\lambda(\Pi)$ by the formula

$$\pi_\lambda(g)f(z) = (cz+d)^{-\lambda} f\left(\frac{az+b}{cz+d}\right)$$

for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Geometric setting

We consider the case $(G, G') = (SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), \Delta SL_2(\mathbb{R}))$.
We are interested in the tensor product representation of two discrete series $\pi_{\lambda'} \otimes \pi_{\lambda''}$ which is considered as a representation of $SL_2(\mathbb{R})$ on the space $H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi)$.

Branching law

$$\pi_{\lambda'} \otimes \pi_{\lambda''}|_{SL_2(\mathbb{R})} \simeq \sum_{l \in \mathbb{N}} \pi_{\lambda'+\lambda''+2l}$$

Multiplicity free branching law

$$\Leftrightarrow$$

dim $Hom_{SL_2(\mathbb{R})}(\pi_{\lambda'} \otimes \pi_{\lambda''}|_{SL_2(\mathbb{R})}, \pi_{\lambda'''}) = 1$

Let $\lambda',\ \lambda'',\ \lambda''' \in \mathbb{N}\backslash\{0;1\}$ such that $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$.
For $f \in H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi)$, we define the Rankin-Cohen brackets as :

Rankin-Cohen bi-differential operators

$$RC^{\lambda'''}_{\lambda',\lambda''}(f)(z) = \sum_{j=0}^{l} \frac{(-1)^j(\lambda' + l - j)_j(\lambda'' + j)_{l-j}}{j!(l-j)!} \frac{\partial^l f}{\partial z_1^{l-j}\partial z_2^j}(z,z)$$

Each operator $RC^{\lambda'''}_{\lambda',\lambda''}$ is a *symmetry breaking operator* from $H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi)$ to $H^2_{\lambda'''}(\Pi)$, and the family of all $RC^{\lambda'''}_{\lambda',\lambda''}$ defines a *symmetry breaking transform*.

Holographic inversion

Relative reproducing kernel

We define $K^{\lambda'''}_{\lambda',\lambda''}$, called the relative reproducing kernel, for $w_2,\ w_1,\ z \in \Pi$:

$$K^{\lambda'''}_{\lambda',\lambda''}(z, w_1, w_2) = (w_2 - w_1)^l \left(\frac{w_1 - \bar{z}}{2i}\right)^{-(\lambda'+l)} \left(\frac{w_2 - \bar{z}}{2i}\right)^{-(\lambda''+l)}$$

Theorem (L.)

Set $\lambda',\ \lambda'',\ \lambda''' \in \mathbb{N}\backslash\{0;1\}$ *such that* $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. *Let* $w_1,\ w_2 \in \Pi$, *and* $g \in H^2_{\lambda'''}(\Pi)$. *Then the adjoint operator* $\left(RC^{\lambda'''}_{\lambda',\lambda''}\right)^*$ *is given by :*

$$(RC^{\lambda'''}_{\lambda',\lambda''})^* g(w_1, w_2) = C \int_{\Pi} g(z) K^{\lambda'''}_{\lambda',\lambda''}(z, w_1, w_2) d\mu(z)$$

And this is an holographic operator.

Set $\lambda',\ \lambda'',\ \lambda''' \in \mathbb{N}\backslash\{0;1\}$ such that $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Define the operator $\Psi^{\lambda'''}_{\lambda',\lambda''}$ on the space $H_{\lambda'''}(\Pi)$ by

$$\Psi^{\lambda'''}_{\lambda',\lambda''}(g)(w_1,w_2) = \frac{(w_1-w_2)^l}{2^{\lambda'+\lambda''+2l-1}l!}\int_{-1}^{1} g(w(v))(1-v)^{\lambda'+l-1}(1+v)^{\lambda''+l-1}dv$$

Theorem (Kobayashi, Pevzner)

The operator $\Psi^{\lambda'''}_{\lambda',\lambda''}$ is an holographic operator from $H_{\lambda'''}(\Pi)$ to $H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi)$.

For $\lambda \in \mathbb{N}\backslash\{0.1\}$, we consider the space $L^2(\mathbb{R}^+, t^{1-\lambda}\ dt) := L^2_\lambda(\mathbb{R}^+)$ Then the Fourier transform defined, for $f \in L^2_\lambda(\mathbb{R}^+)$, by :

$$\mathcal{F}f(z) = \int_{\mathbb{R}^+} f(t)e^{itz}\ dt$$

is an one-to-one isometry (up to a scalar) from $L^2_\lambda(\mathbb{R}^+)$ to $H^2_\lambda(\Pi)$. This allows us to transport the holomorphic discrete series representation of $SL_2(\mathbb{R})$ on this space.

L^2-model

We define the counterpart of $RC^{\lambda'''}_{\lambda',\lambda''}$ in this model thanks to the following commutative diagram

$$
\begin{array}{ccc}
L^2_{\lambda'}(\mathbb{R}^+)\hat{\otimes}L^2_{\lambda''}(\mathbb{R}^+) & \xrightarrow{\widehat{RC^{\lambda'''}_{\lambda',\lambda''}}} & L^2_{\lambda'''}(\mathbb{R}^+) \\
\downarrow{\scriptstyle \mathcal{F}\otimes\mathcal{F}} & & \downarrow{\scriptstyle \mathcal{F}} \\
H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi) & \xrightarrow[RC^{\lambda'''}_{\lambda',\lambda''}]{} & H^2_{\lambda'''}(\Pi)
\end{array}
$$

Symmetry breaking operator in the holomorphic and L^2-model

Theorem (Kobayashi, Pevzner)

Set $\lambda',\ \lambda'',\ \lambda''' \in \mathbb{N}\backslash\{0;1\}$ *such that* $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. *Let* $F \in L^2_{\lambda',\lambda''}(\mathbb{R}^+ \times \mathbb{R}^+)$ *then*

$$\widehat{RC^{\lambda'''}_{\lambda',\lambda''}}F(t) = \frac{t^{l+1}}{2i^l}\int_{-1}^{1} P_l^{\lambda'-1,\lambda''-1}(v)F\left(\frac{t}{2}(1-v), \frac{t}{2}(1+v)\right)\, dv$$

This operator is a *symmetry breaking operator* in the L^2-model.

L^2-model

For the holographic transform we have the following situation

$$\begin{array}{ccc} L^2_{\lambda'}(\mathbb{R}^+)\hat{\otimes}L^2_{\lambda''}(\mathbb{R}^+) & \longleftarrow & L^2_{\lambda'''}(\mathbb{R}^+) \\ \Big\downarrow{\scriptstyle \mathcal{F}\otimes\mathcal{F}} & & \Big\downarrow{\scriptstyle \mathcal{F}} \\ H^2_{\lambda'}(\Pi)\hat{\otimes}H^2_{\lambda''}(\Pi) & \xleftarrow{\Psi^{\lambda'''}_{\lambda',\lambda''}} & H^2_{\lambda'''}(\Pi) \end{array}$$

Holographic operator in the holomorphic and L^2-model

Theorem (Kobayashi, Pevzner)

Set $\lambda',\ \lambda'',\ \lambda''' \in \mathbb{N}\backslash\{0;1\}$ *such that* $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. *We define the following operator which associate to a function* $g(t)$ *defined on* $\mathbb{R}^+$ *a function of two variables on* $\mathbb{R}^+ \times \mathbb{R}^+$ *:*

$$\Phi_{\lambda',\lambda''}^{\lambda'''} g(x,y) = \frac{x^{\lambda'-1} y^{\lambda''-1}}{(x+y)^{\lambda'+\lambda''+l-1}} P_l^{\lambda'-1,\lambda''-1}\left(\frac{y-x}{x+y}\right) g(x+y)$$

Then $\Phi_{\lambda',\lambda''}^{\lambda'''}$ *is an holographic operator from* $L^2_{\lambda'''}(\mathbb{R}^+)$ *to* $L^2_{\lambda',\lambda''}(\mathbb{R}^+ \times \mathbb{R}^+)$

A geometric interpretation

A stratification of the cone $\mathbb{R}^+ \times \mathbb{R}^+$

We use the following diffeomorphism θ from $\mathbb{R}^+ \times (-1,1)$ to $\mathbb{R}^+ \times \mathbb{R}^+$:

$$\theta(t,v) = (\frac{t}{2}(1-v), \frac{t}{2}(1+v))$$

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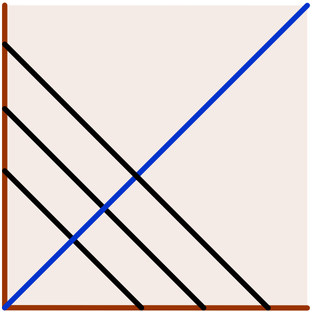
This corresponds to the following stratification of the cone $\mathbb{R}^+ \times \mathbb{R}^+$



A geometric interpretation

Using this diffeomorphism, one can show the following isomorphism

$$\begin{aligned} & L^2_{\lambda',\lambda''}(\mathbb{R}^+ \times \mathbb{R}^+) \\ \simeq\ & L^2\left(\mathbb{R}^+, t^{\lambda'+\lambda''-1}\, dt\right) \hat{\otimes} L^2\left((-1;1), (1-v)^{\lambda'-1}(1+v)^{\lambda''-1}\, dv\right) \end{aligned}$$

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Theorem (L.)

The orthogonal projection on the space $L^2\left(\mathbb{R}^+, t^{\lambda'+\lambda''-1}\, dt\right) \hat{\otimes}\, \mathbb{C} \cdot P_l^{\lambda'-1,\lambda''-1}(v)$ is a symmetry breaking operator for the $SL_2(\mathbb{R})$ representation transported from the L^2-model.

Thank you for your attention.