Algebraic differential operators on unitary groups and their applications

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Abstract

Algebraic differential operators are described acting on automorphic forms φ on unitary groups $U(n,n)$ over an imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-D_{\mathcal{K}}}) \subset \mathbb{C}$. Applications are given to special L-values $L(s,\varphi)$ attached to φ.
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1. The simplest case of modular forms for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.
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3. Unitary groups $U(a, b)$ (of signature (a, b), $a + b = n$), and the double group $U(n, n)$.

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8. Applications to critical values of the standard zeta function $\mathcal{L}(s,\varphi)$ in the unitary case.
9. Perspectives and examples for $U(n,n)$.

Algebraic differential operators in the simplest case of modular forms for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

Action of the derivative $D = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$(where $q = e^{2\pi iz}$) on a modular form $g = \sum_{n=0}^{\infty} b_n q^n$ is not a modular form, but it is quasi-modular ([DZ], p.59, [MaRo], p.67):

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$$(cz+d)^{-\ell-2r} D^r g(\gamma z) = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(t+\ell)} \left(\frac{1}{2\pi i}\frac{c}{cz+d}\right)^{r-t} D^t g(z)$$

for a modular form $g \in \mathcal{M}_\ell(\Gamma)$ of weight ℓ, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

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In order to adjust it to the weight $\ell + 2r$, let us use $S = \frac{1}{4\pi y}$, $\mathrm{Im}\, z = \frac{z - \bar{z}}{2i}$, and $\frac{1}{\mathrm{Im}\,\gamma z} = \frac{|cz+d|^2}{\mathrm{Im}\, z} = (cz+d)\left(-2ic + \frac{cz+d}{y}\right)$:

Maass-Shimura differential operator

If $f = D^r g$ where $g \in \mathcal{M}_\ell(\Gamma)$ is a modular form of weight ℓ, then the transformation law produces also the Maass-Shimura differential operator δ_ℓ to the space of nearly holomorphic forms of weight $\ell + 2r$:

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which preserves the rationality of the coefficients of S and q.

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$\delta_\ell(g) = \frac{1}{2\pi i} y^{-\ell} \frac{\partial}{\partial z}(y^\ell g) = \frac{1}{2\pi i}\left(\frac{\partial g(z)}{\partial z} + \frac{\ell}{2iy} g(z)\right) = (D - \ell S)(g)$

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For an integer $r \geq 0$, $\delta_\ell^r := \delta_{\ell+2r-2} \circ \cdots \circ \delta_\ell$ (see also [U14]).

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A conceptual explanation of the algebraicity comes from the Gauss-Manin connection (due to Grothendieck in higher dimensions see [Gr66], [KaOd68]).

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$\delta_k f(Z) = (-4\pi)^{-n} \det(Z - \bar{Z})^{\frac{1+n}{2} - k} \Delta(\det(Z - \bar{Z})^{k - \frac{1+n}{2} + 1} f(Z))$

acts on q^T via the polynomial representations $\rho_r : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\wedge^r \mathbb{C}^n)$ and its adjoint ρ_r^* (see [CourPa])

$\delta_k(q^T) = \sum_{\ell=0}^{n} (-1)^{n-\ell} c_{n-\ell}(k + 1 - \frac{1+n}{2}) \mathrm{tr}({}^t\rho_{n-\ell}(S) \rho_\ell^*(T)) q^T$, where

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$[(\rho \otimes \tau)(\alpha)(h)](u) := \rho(\alpha) h({}^t\alpha \cdot u \cdot \alpha)$.

Then D_ρ^e equals $(2\sqrt{-1}\pi)^{-e}$ times the (vector-valued) Maass-Shimura differential operator.

From symplectic case (Type C) to unitary case (Type A)

Siegel modular forms of degree n are holomorphic (vector-valued) functions on $\mathbb{H}_n = \{Z = {}^tZ \in \mathbb{C}_n^n, \mathrm{Im}(Z) > 0\}$ (the Siegel space, (Type C) [Shi00]).

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Automorphic forms on unitary groups (Type A) in [Shi00]
$U(a, b)$ (of degree $n = a + b$) $\rightsquigarrow$ the double group $U(n, n)$,
and the corresponding hermitian space of degree n:
$\mathcal{H}_n = \{z \in \mathbb{C}_n^n \mid i(z^* - z) > 0\}$ where $z^* = {}^t\bar{z}$, $x := (z + z^*)/2$ the hermitian part of z, and $y := (z - z^*)/2$ the anti-hermitian part, such that $i(z^* - z)/2 = iy$ is a positive hermitian matrix.

Note that $z = x + iy$, but x, y are not real: for a hermitian matrix h, the real matrices $\dot{h} = \dfrac{\omega\,{}^th - \bar{\omega}h}{\omega - \bar{\omega}}$, $\ddot{h} = \dfrac{h - {}^th}{\omega - \bar{\omega}}$ are used for $\omega = \dfrac{1}{2}(\delta + \delta^{\frac{1}{2}})$, δ the discriminant of $\mathcal{K}$, so that $h = \dot{h} + \omega\ddot{h}$ (notation in [Bra51]).

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$\mathcal{H}_n = \{z \in \mathbb{C}_n^n \mid i(z^* - z) > 0\}$ where $z^* = {}^t\bar{z}$, $x := (z + z^*)/2$ the hermitian part of z, and $y := (z - z^*)/2$ the anti-hermitian part, such that $i(z^* - z)/2 = iy$ is a positive hermitian matrix.

Note that $z = x + iy$, but x, y are not real: for a hermitian matrix h, the real matrices $\dot{h} = \dfrac{\omega\, {}^th - \bar{\omega}h}{\omega - \bar{\omega}}$, $\ddot{h} = \dfrac{h - {}^th}{\omega - \bar{\omega}}$ are used for $\omega = \dfrac{1}{2}(\delta + \delta^{\frac{1}{2}})$, δ the discriminant of $\mathcal{K}$, so that $h = \dot{h} + \omega\ddot{h}$ (notation in [Bra51]).

From symplectic case (Type C) to unitary case (Type A)

Siegel modular forms of degree n are holomorphic (vector-valued) functions on $\mathbb{H}_n = \{Z = {}^tZ \in \mathbb{C}^n_n, \mathrm{Im}(Z) > 0\}$ (the Siegel space, (Type C) [Shi00]).

Automorphic forms on unitary groups (Type A) in [Shi00]
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Automorphic L functions on unitary groups and related geometric objects
where discussed by M. Harris (ICM 2014), *Automorphic Galois representations and the cohomology of Shimura varieties.*, [Ha14], and by P.Scholze (ICM 2018), Applications of p-adic geometry to automorphic Galois representations on unitary groups in [Scho18].

Unitary groups and forms, [Ha97],[EE], [Shi00]

Unitary groups $U(a, b)$ $(a + b = n)$ and $U(n, n)$ (the double group). Let V be an n-dimensional space over an imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-D_{\mathcal{K}}})$, and let $\langle \cdot, \cdot \rangle$ be a non degenerate hermitian pairing of signature (a, b) on V relative to $\mathcal{K} \subset \mathbb{C}$.

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For a vector space W with hermitian pairing $\langle\cdot,\cdot\rangle_W$, and a $\mathbb{Q}$-algebra R, the unitary groups are defined by
$U(W)(R)=\left\{g\in\mathrm{GL}(W\otimes R)|\forall v,v',\langle gv,gv'\rangle=\langle v,v'\rangle\right\}$
$GU(W)(R)=\left\{g\in\mathrm{GL}(W\otimes R)|\forall v,v',\exists\nu(g)\in R^*,\langle gv,gv'\rangle=\nu(g)\langle v,v'\rangle\right\}.$
Then $U(2V)(R)\cong U(n,n)(R)$
$=\left\{M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{GL}_{2n}(\mathcal{K}\otimes R)|M\eta_nM^*=\eta_n\right\},\ \eta_n=\begin{pmatrix}0_n&-I_n\\I_n&0_n\end{pmatrix}.$

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$=\left\{M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{GL}_{2n}(\mathcal{K}\otimes R)|M\eta_nM^*=\eta_n\right\}$, $\eta_n=\begin{pmatrix}0_n&-I_n\\I_n&0_n\end{pmatrix}$. The group $U(n,n)$ acts on the hermitian space
$\mathcal{H}_n=\{z\in\mathbb{C}_n^n\mid i(z^*-z)>0\}$, where $z^*:={}^t\bar{z}$.

Algebraic geometric approach: families of abelian varieties of CM-type and unitary groups

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Main arithmetical applications of unitary groups $U(n, n)$ use Shimura's analytic families of abelian varieties A of CM-type of $\dim_{\mathbb{C}} A = 2n$, that is, with fixed imbedding $\iota : \mathcal{K} \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$, and other PEL-structures ("polzarization, endomorphisms, level", following [EE], §2).

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Recall that elliptic curves E with complex multiplication by $\mathcal{K}$ correspond to certain CM-points on the upper half plaine $\mathbb{H}$, that is $E \xrightarrow{\sim} \mathbb{C}/L$, where $L = \langle 1, \alpha \rangle \subset \mathcal{K} = \mathbb{Q}(\alpha)$ is a lattice in $\mathbb{C}$ and $\mathrm{Im}\,(\alpha) > 0$ (only special CM-points, not analytic families).

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Families of $2n$-dimensional CM-abelian varieties A use the analytic parameter $z \in \mathcal{H}_n$. Any row vector $x \in \mathcal{K}^1_{2n}$ defines a z-holomorphic $\mathbb{C}^{2n}$-valued function $p_z(x)$ by

$$p_z(x) = \left([z, 1_n] \cdot x^*, [{}^t z, 1_n] \cdot {}^t x\right)$$

For a fixed lattice $L \subset \mathcal{K}^{2n} \subset \mathbb{C}^{2n}$, denote by $L_z = p_z(L)$ a $4n$-dimensional CM-lattice of analytic parameter z.

Explicit matrix description by the complex torus $\mathbb{C}^{2n}/L_z$

Any $2n$-dimensional abelian variety of CM-type is isomorphic to A_z, with the action of $\mathcal{K}$ given by $\iota_z(a) \cdot v = \mathrm{diag}[\bar{a} \cdot 1_n, a \cdot 1_n] \cdot v$.

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Universal analytic family $\mathcal{A}_{univ}$ over $\mathcal{H}_n$: taking L the lattice in $\mathcal{K}^1_{2n}$ generated by the standard basis vectors $e_1, \cdots, e_{2n}$, and the vectors $\alpha\cdot e_1, \cdots, \alpha\cdot e_{2n}$ with α a generator of $\mathcal{K}$ over $\mathbb{Q}$. Then the fiber A_z over each point $z = (z_{ij}) \in \mathcal{H}_n$ is the abelian variety $A_z \cong \mathbb{C}^{2n}/L$, where L_z the $\mathbb{Z}$-lattice generated by $4n$ rows:

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$$
\begin{aligned}
z_j &= (z_{1j}, \cdots, z_{nj}, z_{j1}, \cdots, z_{jn}) \\
e_j &= \text{ vector with 1 in the } j\text{-th and } j+n\text{-th positions} \\
&\quad \text{and zeroes everywhere else,} \\
z'_j &= (\bar{\alpha} z_{1j}, \cdots, \bar{\alpha} z_{nj}, \alpha z_{j1}, \cdots, \alpha z_{jn}) \\
e'_j &= \text{ vector with } \bar{\alpha} \text{ in the } j\text{-th, and } \alpha \text{ in the } j+n\text{-th positions} \\
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Vector-valued automorphic forms on unitary groups

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Weight ρ of an automorphic form on G is a representation of the maximal compact subgroup $K \subset G$. Weights are constructed via the following polynomial representations $\rho_{\boldsymbol{\kappa}} : \mathrm{GL}_n \to \mathrm{GL}(V_{\boldsymbol{\kappa}})$.
For each set $\boldsymbol{\kappa}$ of orderered integers $\kappa_1 \geq \cdots \geq \kappa_n$ there is a representation $(\rho_{\boldsymbol{\kappa}}, \mathrm{GL}_n)$ of highest weight $\boldsymbol{\kappa}$, constructed as
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Vector valued modular forms $\mathcal{M}_{\boldsymbol{\kappa}}$ (symplectic case) and $\mathcal{M}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}$ (unitary case) can be attached to the representations with highest weight $\rho = \rho_{\boldsymbol{\kappa}}$ and $\rho^+_{\boldsymbol{\kappa}} \otimes \rho^-_{\boldsymbol{\kappa}'}$ of the maximal compact subgroups $K \cong U(n) \subset Sp_{2n}(\mathbb{R})$ and $K \cong U(a) \times U(b) \subset U(a,b)$.

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These modular forms take values in $V_{\boldsymbol{\kappa}}$ and $V_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}$, and defined on the symmetric spaces G/K, $G = Sp(\mathbb{R})$ or $G = U(a,b)$.
Some notation $\alpha(z) = (az+b)(cz+d)^{-1}$, $\lambda(z) = \bar{c} \cdot {}^t\bar{z} + \bar{d}$, $\mu(z) = c \cdot z + d$ (used for the automorphy factors of weight ρ, and for the Eisenstein series).

C^∞-differential operators via Shimura's approach

For each $z \in \mathcal{H}_n$, let $\Xi(z) = (\xi(z), \eta(z)) = (i(\bar{z} - {}^t z), i(z^* - z))$, so that ${}^t\xi(z) = \eta(z) = i(z^* - z))$. The tangent space $T = \mathbb{C}_n^n$ over $\mathbb{C}$ has a $\mathbb{R}$-rational basis $\{e_\nu\}$, $u := \sum_\nu u_\nu \varepsilon_\nu$, $z := \sum_\nu z_\nu \varepsilon_\nu$.

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Let $(\rho, V) = (\rho_- \otimes \rho_+, V_- \otimes V_+)$ be a finite dimensional representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, and e be a positive integer. For vector spaces X and Y, define $S_e(Y, X)$ the vector space of degree e homogeneous polynomial maps of Y into X, i.e. the space of maps h from Y to X such that $h(a \cdot y) = a^e h(y)$, $S_e(Y) = S_e(Y, \mathbb{C})$.

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$$(Df)(u) = \sum_\nu u_\nu \frac{\partial f}{\partial z_\nu}$$
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For $e > 1$ write $D^e(f)$ and $C^e(f)$ for $D(D^{e-1}f)$ and $C(C^{e-1}f)$, viewed as $C^\infty(\mathcal{H}_n, S_e(T, V))$ - valued.

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Action on vector-values automorphic forms

Given $g = (a, b) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, (ρ, X) a polyomial representation, and $h \in M\ell_e(T, X) = M\ell_e(T, \mathbb{C}) \otimes X$ (symmetric $\mathbb{R}$-multilinear map viewed also as element $S_e(T, X)$), define
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$$[(\rho \otimes \tau^e)(g)](h(u) \otimes x) = \tau^e(g)h \otimes \rho(g)x$$

for each $g \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, $h \in M\ell_e(T, \mathbb{C})$, and $x \in X$. For $e > 1$ write $D^e(f) = D(D^{e-1}(f))$ and $C^e(f) = C(C^{e-1})(f)$.

Action on vector-values automorphic forms

Given $g = (a, b) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, (ρ, X) a polyomial representation, and $h \in M\ell_e(T, X) = M\ell_e(T, \mathbb{C}) \otimes X$ (symmetric $\mathbb{R}$-multilinear map viewed also as element $S_e(T, X)$), define $[\tau^e(a, b)h](u_1, \cdots, u_e) = ({}^t a u_1 b, \cdots, {}^t a u_e b)$, and a representation $\rho \otimes \tau^e$ of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on $M\ell_e(T, \mathbb{C}) \otimes X$

$$[(\rho \otimes \tau^e)(g)](h(u) \otimes x) = \tau^e(g)h \otimes \rho(g)x$$

for each $g \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, $h \in M\ell_e(T, \mathbb{C})$, and $x \in X$. For $e > 1$ write $D^e(f) = D(D^{e-1}(f))$ and $C^e(f) = C(C^{e-1})(f)$.

Such operators take automorphic forms of weight ρ to automorphic forms of weight $\rho \otimes \tau^e$ as follows: define

$$(D_\rho f)(u) = \rho(\Xi)^{-1} D[\rho(\Xi)f](u) = (\rho \otimes \tau)(\Xi)^{-1} C[\rho(\Xi)f](u).$$

and $(D^e_\rho f)(u) = (\rho \otimes \tau^e)(\Xi)^{-1} C^e[\rho(\Xi)f]$ for $e > 1$.
Then D^e_ρ maps automorphic forms of weight ρ to automorphic forms of weight $\rho \otimes \tau^e$.

General Shimura's differential operators D_ρ^Z via φ_Z

The classification of the irreducible subspaces of polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ and of irreducible subspaces of τ^e is studied in [Shi00], Theorem 12.7, in terms of highest weights. Given a matrix $a \in \mathbb{C}_n^n$, let $\det_j(a)$ denote the determinant of the upper left $j \times j$ submatrix of a. If ρ and σ are irreducible representations of $\mathrm{GL}_n(\mathbb{C})$, $\rho \otimes \sigma$ occurs in τ^e if and only if

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$$\prod_{j=1}^{n} \det_j(x)^{e_j} \ (x \in T = \mathbb{C}_n^n, e_j = \kappa_j - \kappa_{j+1}, 1 \leq j \leq n-1, e_n = \kappa_n)$$

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If ρ is the representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, there is a differential operator D^Z defined for a stable quotient of $S_e(T)$ with the projection φ_Z of $S_r(T) \otimes X$ onto $Z \otimes X$. Then the operator $D^Z_\rho = \varphi_Z D^e_\rho$ is a map from the space of automorphic forms of weight ρ to automorphic forms of weight $\rho \otimes \tau_Z$, where τ_Z denotes the restriction of τ to Z. There is a formula for the action of the algebraic differential operators θ^Z_ρ on formal q-expansions on the double group G at a at a cusp (which is a certain formal object) $f = \sum_{L_m \ni \boldsymbol{\beta} > 0} a(\boldsymbol{\beta}) q^{\boldsymbol{\beta}}$, where L_m is the lattice in $Herm_{\mathcal{K}}$ determined by m.

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Algebraic differential operators

on automorphic forms on unitary groups. Fix a $\mathcal{O}_{\mathcal{K}}$-algebra $\mathcal{R}$ with inclusion $\iota : \mathcal{R} \to \mathbb{C}$ and a weight representation $\rho = (\rho^+, \rho^-)$ of the maximal compact subgroup $K = U(n) \times U(n)$ of $U = U(n, n)$.

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Then a general algebraic operator $\theta(f)$ is defined as above via $\theta(\zeta)(f)$, using $\boldsymbol{\beta}$ and Ξ as formal variables over a cusp:
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$\theta(\zeta)(f)(q) = \sum_{\beta \in H_{\geq 0}} \zeta(\boldsymbol{\beta}) c_{\boldsymbol{\beta}}(\Xi) q^{\boldsymbol{\beta}}$
This construction allows to treat vector-valued modular forms as polynomial-valued, and to prove congruences between them monomial-by-monomial.

Classical setting: arithmetic differential operators

In the Unitary case such operators were studied in [EE]; we may write $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$ in the q expansion on the double group, with hermitian matrices β_1, β_4 , and $\beta_2^* = \beta_3$. In the Sp-case such operator studied in [BS00] and [Do17] are compositions Shimura-type operators, described then via its action on the q-expansions.
For $\nu \in \mathbb{N}$, we put

$$\mathfrak{D}^{\nu}_{n,\alpha} = \mathfrak{D}_{n,\alpha+\nu-1} \circ \ldots \circ \mathfrak{D}_{n,\alpha}$$
$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha} = (\mathfrak{D}^{\nu}_{n,\alpha}) \mid_{z_2=0} .$$

The arithmetic applications of this differential operator is due to its explicit action on the exponentials in the Fourier expansion as follows: for $\mathfrak{T} \in \mathbb{C}^{2n,2n}_{\mathrm{sym}}$, we recall a polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(\mathfrak{T})$ defined by S. Böcherer in the entries $t_{ij}(1 \leq i \leq j \leq 2n)$ of $\mathfrak{T}$ by

$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}(e^{\mathrm{tr}(\mathfrak{T}Z)}) = \mathfrak{P}^{\nu}_{n,\alpha}(\mathfrak{T}) e^{\mathrm{tr}(\mathfrak{T}_1 z_1 + \mathfrak{T}_4 z_4)}, \mathfrak{T} = \begin{pmatrix} \mathfrak{T}_1 & \mathfrak{T}_2 \\ {}^t\mathfrak{T}_2 & \mathfrak{T}_4 \end{pmatrix}, \mathfrak{Z} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

that is, it represents "action of differential operator on exponential function". The $\mathfrak{P}^{\nu}_{n,\alpha}$ are homogenous polynomials of degree $n\nu$.

Applications to critical values

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More generaly, take a unitary group U of a n-dimensional $\mathcal{K}$-vector space with a non-degenerate hermitian form $\langle \cdot, \cdot \rangle_V : V \times V \to \mathcal{K}$of signature (a, b), $a + b = n$. Then a vector-valued automorphic (Hecke eigenform) φ on U generates a cuspidal automorphic representation $\pi = \pi_\varphi$ of the adelic group $U(\mathbb{A})$.

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The standard zeta function $L(\varphi,\chi,s)=L(\pi_\varphi,\chi,s)$ with a Hecke character $\chi:\mathbb{A}_{\mathcal{K}}^\times\to\mathbb{C}^\times$ of allowed type χ_∞ is a certain Euler product $L(\varphi,\chi,s)=\prod_{\mathfrak{q}} L_{\mathfrak{q}}(\varphi,\chi,s)$, where $L_{\mathfrak{q}}(\varphi,\chi,s)^{-1}=L_{\mathfrak{q}}(\varphi,X)$ is a polynomial of deg $=2n$ of $X=N(\mathfrak{q})^{-s}\chi(\mathfrak{q})$ given by the Satake parameters $t_{\mathfrak{q},i}$ $(i=1,\ldots,n)$ of $\pi_{\mathfrak{q},\varphi}$ (for $\mathfrak{q}$ outside a finite set S). The signature (a,b) is such that $n=a+b$ and $s=\frac{n-1}{2}$ is critical for the L-function $L(\pi,\chi,s)=L(\pi_\varphi,\chi,s)$.

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$$= Z_S(s)L^S(\pi_\varphi, \chi, s + \frac{n-1}{2})\langle\varphi_1, \varphi_2\rangle$$

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where $E((g_1, g_2), f_{s,\chi})$ denotes the restriction to $(g_1, g_2)$of an Eisenstein series on the double adelic group $G = U(a+b, a+b)$, the series defined from a suitably chosen section $f = f_{s,\chi} \in \mathrm{Ind}^G_{P_{Siegel}}$, $\varphi_1 \in \pi, \varphi_2 \in \tilde{\pi}$, with

$P_{Siegel} = \begin{pmatrix} * & * \\ 0_{a+b} & * \end{pmatrix}$ is the Siegel parabolic in G,

$$E(g, f) = \sum_{\gamma\in P(\mathcal{K})\backslash G(\mathcal{K})} f(\gamma g), \ \ f_{k,\chi} = \chi(\det(c))\det(cz+d)^{-k},$$

$$\langle\varphi_1, \varphi_2\rangle = \int_{U(a,b)} \varphi_1(g)\varphi_2(g)dg.$$

For the critical values $s = s_*, \ldots, s^*$ we use certain algebraic operators θ_{s^*-s} to move the Eisenstein series from s^* to s by acting on the section $f_{s^*,\chi}$ to get $f_{s,\chi}$. This allows to compare their q-expansions and get congruences for the critical values.

Classical setting: pull-back identity

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$$\mathcal{F}(g) = \frac{\left\langle \left\langle \boldsymbol{f}_1^0(w), g(*,*) \right\rangle^w, \boldsymbol{f}_2^0(z) \right\rangle^z}{\langle \boldsymbol{f}_1^0, \boldsymbol{f}_2^0 \rangle}$$

From test functions $g = g_{\chi_i, s_i}(*,*)$ to normalized critical L-values $\mathcal{D}(\boldsymbol{f}, t_i, \chi_i) = \mathcal{F}(g_{\chi_i, s_i}) = L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$ at t_i with $k_i + t_i = \ell$
Here $g(z, w) = \mathcal{H}_{t,\chi}(-\bar{z}, w)$ is a function in the tensor product of certain spaces of automorphic forms

$$\mathcal{H}_{t,\chi} \in C^\infty M_n^\ell(\Gamma_0(M), \varphi)|_z \otimes_{\mathbb{C}} C^\infty M_n^\ell(\Gamma_0(M), \varphi)|_w,$$

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obtained from a double Eisenstein series E_{k_i, χ_i} on $U(n, n)$ of the above type, with $\boldsymbol{f}_1^0$, $\boldsymbol{f}_2^0$ suitably chosen eigenfunctions of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ (the Hermitian Fourier expansion): .

Eisenstein series and congruences (Unitary case)

(KEY POINT!) The (Siegel-Hermite) Eisenstein series $E_{2\ell,n,K}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined in [Ike08] by
$E_{2\ell,n,K}(Z) = \sum_{g\in\Gamma_{n,K,\infty}\backslash\Gamma_{n,K}} (\det g)^{\ell} j(g,Z)^{-2\ell}$ (converges for $\ell > n$). The normalized Eisenstein series is given by
$\mathcal{E}_{2\ell,n,K}(Z) = 2^{-n}\prod_{i=1}^{n} L(i-2\ell,\theta^{i-1})\cdot E_{2\ell,n,K}(Z)$.

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If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is polynomial over $\mathbb{Z}$ in variables $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here, $\tilde{F}_p(H,X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a key point in proving congruences for the modular forms in both the pull-back double integral representation and Rankin-Selberg integral.

Strategy of the construction of p-adic L-functions

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- The integral representation for the normalized critical values $L^*(\boldsymbol{\pi}, \chi_i, s_i,)$ via the doubling method:
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  $$= \int\limits_{U \times U_-} E((g_1, g_2), f_{s_i,\chi_i})\chi_i^{-1}(\det g_2)\boldsymbol{\varphi}_{i,1}(g_1)\boldsymbol{\varphi}_{i,2}(g_2)dg_2$$
  where $\boldsymbol{\varphi}_{i,1} \in \boldsymbol{\pi}, \boldsymbol{\varphi}_{i,1} \in \tilde{\boldsymbol{\pi}}$ are chosen functions in dual spaces (factorizable adelic Schwartz functions on the group $U(n)(\mathbb{A})$), $E((g_1, g_2), f_{s_i,\chi_i})$ the pull-back of the Eisenstein series on $U(n,n)$, $f = f_{s_i,\chi_i}$ its Siegel section $f \in I_P^U = \mathrm{Ind}_{P_{Siegel}}^{U(n,n)}$,
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where $\boldsymbol{\varphi}_{i,1} \in \boldsymbol{\pi}, \boldsymbol{\varphi}_{i,1} \in \tilde{\boldsymbol{\pi}}$ are chosen functions in dual spaces (factorizable adelic Schwartz functions on the group $U(n)(\mathbb{A})$), $E((g_1, g_2), f_{s_i,\chi_i})$ the pull-back of the Eisenstein series on $U(n, n)$, $f = f_{s_i,\chi_i}$ its Siegel section $f \in I_P^U = \mathrm{Ind}_{P_{Siegel}}^{U(n,n)}$,
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- From Siegel sections f_{χ_i,s_i} to critical values $L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$. Families of automorphic distributions $\{\boldsymbol{\mu}_r\}$, $0 \le r \le s^* - s_*$ on the weight space X attached to $U(a, b)$. They produce $\bar{\mathbb{Q}}$-valued distributions $\boldsymbol{\mu}_i$ on X such that $\int_X \chi_i(x_p)d\boldsymbol{\mu}_{s^*-s_i} = L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$, where $X_\pi \to \mathbb{Z}_p^*$ is a p-part projection. Fixing embeddings $\bar{\mathbb{Q}} \overset{i_\infty}{\hookrightarrow} \mathbb{C}$, $\bar{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ produces p-adic-valued distributions.

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  Take a dense family of continuous functions $\{\varphi_i = \varphi_{s_i,\chi_i}\}$ in $\mathcal{C}(X_\pi, \mathbb{C}_p)$ on the p-adic space X_π. Then Kummer says:
  $\sum_i \beta_i \varphi_i \equiv 0 \bmod p^N \Longrightarrow \sum_i \beta_i L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i) \equiv 0 \bmod p^N$.
  Each $\varphi \in \mathcal{C}(X_\pi, \mathbb{C}_p)$ can be approximated by $\{\varphi_i\}_i$, and a measure $\mu_{\boldsymbol{\pi}}(\varphi)$ with given $\mu_{\boldsymbol{\pi}}(\varphi_i) = L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$ is a well-defined limit over all approximations of φ.

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- From bounded measures on X to admissible measures using $h_{\boldsymbol{\pi},p} = P_{Newton,p}(d/2) - P_{Hodge}(d/2) \geq 0$.
  Computing critical values at $s = s_*, \cdots, s^*$ and prove admissibility congruences for them as follows
  A $\mathbb{C}_p$-linear mapping $\mu : \mathcal{C}^h \to M$ is called an h admissible M-valued measure on $\mathbb{Z}_p^*$ if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

  for $j = 0, 1, ..., h-1$. Such μ extends to $\mathcal{C}^{loc-an}$ (and to $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$, the space of definition of p-adic Mellin transform)

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The Hodge polygon of (weak) pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \le \frac{w}{2}} \Gamma_{j,w-j}(s)$, where

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The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \ldots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$:
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$P_{Newton,p}(t) \ge P_{Hodge}(t)$ on $[0, d]$, see also [BrCo].

Hodge/Newton polygons for $\mathbf{f} = Lift(\Delta), n = 3,\ U(3,3)$

Let us draw $P_{Hodge}(t)$ (slopes 0,1,2,11,12,13), and $P_{Newton,p}(t)$ (slopes 1,2,3,10,11,12), symmetry for slopes: $j \mapsto 13 - j$, for $p = 7$, $\boldsymbol{f} = Lift(\Delta)$,
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Let us draw $P_{Hodge}(t)$ (slopes 0,1,2,11,12,13), and $P_{Newton,p}(t)$ (slopes 1,2,3,10,11,12), symmetry for slopes: $j \mapsto 13 - j$, for $p = 7$, $\boldsymbol{f} = Lift(\Delta)$, $k = 12$, $n' = 1$, $\ell = 14 = k + 2n'$, $d = 4n = 12$,
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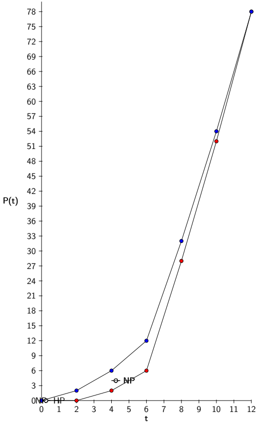

$P_{Newton,p}(6) = 12$, $P_{Hodge}(6) = 6$, $h = 6$ ("the Hasse invariant")

Description of the Main theorem

Let $\Omega_{\mathbf{f}}$ be a period attached to an Hermitian cusp eigenform $\mathbf{f}$,
$\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p}\prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ (the Hermitian Fourier expansion) on some $\mathbf{f_0}$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$, $d = 4n$, $\dfrac{d}{2} = 2n$.

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Definition. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$:
$\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then
- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) measure M-valued measure on $\mathbb{Z}_p^*$.
- $\mu : \mathcal{C}^h \to M$ is called an h admissible measure M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

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Theorem ([Am-V], [MTT]) For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

Main Theorem.

Let **f** be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 2n$.

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i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s, \chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s, \chi) = \prod_{\mathfrak{q}|p} A_{\mathfrak{q}}(s, \chi)$ including a finite Euler product, Satake parameters $t_{\mathfrak{q},i}$, gaussian sums, the conductor of χ; the integral is a finite sum.

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(ii) if $\mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}) p^{-n(n+1)} \right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their Mellin transforms $\mathcal{L}_{\mu_{\mathcal{D},s}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D},s}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$,
give bounded p-adic analytic interpolation of the above L-values to on the $\mathbb{C}_p$-analytic group $\mathcal{Y}_p$; and these distributions are related by:
$\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} \mu_{\mathcal{D},s^*}$, $X = \mathbb{Z}_p^*$, where $s^* = \ell - n$, $s_* = n$.

Main theorem (continued)

(iii) in the admissible case assume that $0 < h \leq s^* - s_* + 1 = \ell + 1 - 2n$, where $h = \mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}) p^{-n(n+1)} \right) > 0$, Then there exists an h–admissible measure $\mu_{\mathcal{D}}$ whose integrals $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ are given by $i_p \left(A_p(s,\chi) \dfrac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}} \right) \in \mathbb{C}_p$ with $A_p(s,\chi)$ as in (i); their Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$.

Main theorem (continued)

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(iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by (i)-(iii).

Remarks.

(a) Interpretation of s^*: the smallest of the "big slopes" of P_H

(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H.

Thanks for your attention!

Appendix A. Recovering geometric objects from automorphic forms and special functions

For an irreducible automorphic representation $\pi = \pi_\varphi$ of a $\mathbb{Q}$-algebraic group $G(\mathbb{A})$, the eventual geometric type of π is determined by the component π_∞, where $\pi = \otimes_v \pi_v$, v the set of valuations.

- (Wiles) Elliptic curves $E/\mathbb{Q} \leftrightarrow$ Hecke cusp eigenforms $f = \sum_{n=1}^\infty a_n q^n$ of weight $w = 2$ and $a_n \in \mathbb{Q}$ (where $q = e^{2\pi i z}$).
- (Deligne,Serre, Scholl, Carayol) Holomorphic modular forms of higher weight $w \geq 2 \rightsquigarrow X_f$, certain $(w-1)$-dimensional parts X_f (called "motives") of a Kuga-Sato variety E_{univ}^{w-2}, such that $L_f(s) = \sum_{n=1}^\infty a_n n^{-s} = L(H^{w-1}(X_f), s)$
- (Manin-Shimura-Mazur) Periods and modular symbols $\int_x^{i\infty} f(z) z^r dz$ $\rightsquigarrow$ Normalized special values $L_f^*(r+1, \chi)$, where $L_f^*(s, \chi) := \Gamma(s) L_f(s, \chi)$, for any Dirichlet character χ, $0 \leq r \leq w-2, x \in \mathbb{Q})$. That is, the integrals on the left give linear forms on homology classes of geodesics $\{x, i\infty\}$, i.e. elements of certain cohomology groups $H^{w-1}(X_f)$, producing X_f and $L(X_f), s)$.
- The use of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]] = Dist(\mathbb{Z}_p, \mathbb{Z}_p)$,

  $$\Lambda \ni \mu \longleftrightarrow A_\mu(T) = \sum_{k \geq 0} A_k T^k, \text{ where } A_k = \int_{\mathbb{Z}_p} \binom{x}{k} d\mu.$$

  The integral $I = \int_{\mathbb{Z}_p} \varphi(x) d\mu(x)$ of any continuous function

  $$\varphi = \sum_{k \geq 0} a_k \binom{x}{k} \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p) \text{ becomes } I = \sum_{k \geq 0} a_k A_k.$$

Appendix B.Prisms and Prismatic cohomology [BhSc19]

This new tool in the theory of geometric p-adic Galois representations appeared since [Scho15], [Scho18] and can be used for the study of q-universal deformation the De Rham cohomology of locally-symmetric hermitian spaces (or Shimura varieties of PEL-type). The above example of unitary groups $U_{\mathcal{K}}(n,n)$ describes analytic families of abelian varieties A with imbedding $\iota : \mathcal{K} \hookrightarrow \mathrm{End}_{\mathcal{K}}(A)$. Thus obtained p-adic schemes $X_{\pi,p}$ produce de Rham cohomology groups as above, and their universal deformations can be described using prisms [BhSc19] as cerain Iwasawa-type modules, notably, $\mathbb{Z}_p[[q-1]]$-modules, where $T = q-1$ is the Iwasawa variable attached to the quantum variable q.
According to [BhSc19], the notion of a prism substitutes in applications the notion of a perfectoid ring. Using prisms, one may attach a ringed site - the prismatic site - to a formal $\mathbb{Z}_p$-scheme. The resulting cohomology theory specializes to most known integral p-adic cohomology theories (étale, crystalline, de Rham). As application, a co-ordinate free description of q-de Rham cohomology is given.
Given a formally smooth $\mathbb{Z}_p$-scheme X, this cohomology yields a deformation of the de Rham cohomology of $X/\mathbb{Z}_p$ across the map $\mathbb{Z}_p[[q-1]] \xrightarrow{q\to 1} \mathbb{Z}_p$.

Appendix C. Ikeda's lifting $f \rightsquigarrow \mathbf{f} = Lift(f)$

Its L-function gives a crucial motivation for both complex and p-adic theory of L-functions on unitary groups, and extends to a general (not necessarily lifted) case. Recall that in [Ike08]

$$S_{2k+1}(\Gamma_0(D), \theta) \ni f \rightsquigarrow \mathbf{f} = Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}), \text{ if } n = 2n' \text{ is even } (E)$$

$$S_{2k}(\mathrm{SL}(\mathbb{Z})) \ni f \rightsquigarrow \mathbf{f} = Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}), \text{ if } n = 2n'+1 \text{ is odd } (O)$$

the standard L-function of $\mathbf{f} = Lift^{(n)}(f)$ is a nice product: $\mathcal{Z}(s, \mathbf{f}) =$

$$\prod_{i=1}^{n} L(s+k+n'-i+(1/2), f)L(s+k+n'-i+(1/2), f, \theta) \quad \text{[Ike08]}$$

$$= \prod_{i=0}^{n-1} L(s+\ell/2-i-(1/2), f)L(s+\ell/2-i-(1/2), f, \theta).$$

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$$= \prod_{i=0}^{n-1} L(s+\ell/2-i-(1/2), f)L(s+\ell/2-i-(1/2), f, \theta).$$

Notice $k+n' = \ell/2$, then the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1-s$ becomes
$\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$.

Appendix D. Special hypergeometric motives and their L-functions: Asai recognition, [DPVZ]

The generalized hypergeometric functions are a familiar player in arithmetic and algebraic geometry. They come quite naturally as periods of certain algebraic varieties, and consequently they encode important information about the invariants of these varieties.

Euler factors, Newton and Hodge polygons attached to them, provide a tool for their geometric recognition.

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