

Formal solution of $\hbar$ – KP hierachy

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Tau-function

Hereafter we work with the $\hbar$ – KP hierarchy from K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math.Phys. **7** (1995) 743-808. Tau-function of the hierarchy is a function $\tau = \tau(\mathbf{t})$ depended on the infinite set of time variables $\mathbf{t} = \{t_1, t_2, \dots\}$. There is also contains $\hbar$ as a parameter but we will not write it explicitly. Below we use the notation

$$\tau^{[z_1, \dots, z_m]}(\mathbf{t}) = e^{\hbar(D(z_1) + \dots + D(z_m))} \tau,$$

where

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k; \quad \partial_k = \partial / \partial t_k. \quad (1)$$

Hirota relation

Let $\hbar > 0$. A function τ is called τ -function of $\hbar$ - KP hierarchy if for any z_1, z_2, z_3 it satisfies $\hbar$ -Hirota functional relation.

$$(z_1 - z_2)\tau^{[z_1, z_2]}\tau^{[z_3]} + (z_2 - z_3)\tau^{[z_2, z_3]}\tau^{[z_1]} + (z_3 - z_1)\tau^{[z_3, z_1]}\tau^{[z_2]} = 0. \quad (2)$$

For $\hbar = 1$ it gives the ordinary Hirota relation.

Theorem

1. A function τ is τ -function of $\hbar$ - KP hierarchy if and only if for any z_1, z_2

$$\hbar \partial_1 \log \frac{\tau^{[z_1]}}{\tau^{[z_2]}} = (z_2 - z_1) \left(\frac{\tau^{[z_1, z_2]}\tau}{\tau^{[z_1]}\tau^{[z_2]}} - 1 \right). \quad (3)$$

2. A function τ is τ -function of $\hbar$ - KP hierarchy if and only if

$$\prod_{1 \leq i < j \leq m} (z_j - z_i) \cdot \tau^{[z_1, \dots, z_m]}\tau^{m-1} = \det_{1 \leq j, k \leq m} \left((z_j - \hbar \partial_1)^{k-1} \tau^{[z_j]} \right) \quad (4)$$

for any $m \geq 2$ and any $z_1, \dots, z_m$.

Schur polynomials

For description of τ -function we use Schur polynomials. An elementary Schur polynomial $h_k(\mathbf{t})$ depends from natural number k and infinite number of variable $\mathbf{t} = (t_1, t_2, \dots)$. It is defined by the generating series

$$\exp\left(\sum_{k \geq 1} t_k z^k\right) = \sum_{k \geq 0} h_k(\mathbf{t}) z^k.$$

An general Schur polynomial $s_\lambda(t)$ depends from a Young diagram $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$ of degree $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$ with a $\ell = \ell(\lambda) \geq 0$ rows of positive lengths $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$.

The general Schur polynomial is the determinant

$$s_\lambda(\mathbf{t}) = \det_{i,j=1,\dots,\ell(\lambda)} h_{\lambda_i - i + j}(\mathbf{t}).$$

Formal tau-function

Let us introduce a deformation $\partial_k^{\hbar}$ of the $\partial_k = \partial/\partial t_k$

$$\partial_k^{\hbar} = \sum_{l=1}^k \frac{\hbar^{l-1} k}{l!} \sum_{\substack{k_1, \dots, k_l \geq 1 \\ k_1 + \dots + k_l = k}} \frac{\partial_{k_1} \dots \partial_{k_l}}{k_1 \dots k_l} = \partial_k + \hbar \sum_{l=1}^{k-1} \frac{k \partial_l \partial_{k-l}}{2l(k-l)} + O(\hbar^2). \quad (5)$$

In the KP theory the first variable t_1 is distinguished. Thus we will consider τ -function as evolution from a function $\tau(x, 0)$ of one variable by $\hbar$ - KP flows

$$\tau(x, 0) \rightarrow \tau(x, \mathbf{t}) = f(x) \hat{\tau}(x + t_1, t_2, t_3, \dots).$$

A formal τ -function of $\hbar$ - KP hierarchy we mean a formal series for a function $\tau(x, \mathbf{t})$ of this type, that satisfy the Hirota equation by $\mathbf{t}$ for any x .

Our first goal is a construction of formal τ -functions of $\hbar$ - KP hierarchy from f , $\tau(x; \mathbf{0})$, and Cauchy-like data

$$\partial_k^{\hbar} \tau(x; \mathbf{t}) \Big|_{\mathbf{t}=\mathbf{0}}.$$

Theorem

Let $\hbar \neq 0$, and $f(x)$, $c_k(x)$, $k = 0, 2, 3, \dots$, be arbitrary infinitely differentiable functions of x (with $c_0(x)$ being not identically 0). Put $c_\emptyset(x) = c_0(x)$ and

$$c_\lambda(x) = (c_0(x))^{1-\ell(\lambda)} \det_{1 \leq i, j \leq \ell(\lambda)} \left[\sum_{k=0}^{j-1} (-\hbar)^k C_{j-1}^k \partial_x^k c_{\lambda_i - i + j - k}(x) \right] \quad (6)$$

for Young diagram $\lambda \neq \emptyset$. Then the series

$$\tau(x; \mathbf{t}) = \sum_{\lambda} c_\lambda(x) s_\lambda(\mathbf{t}/\hbar) \quad (7)$$

is a formal solution to the $\hbar$ -KP hierarchy, where

$$\tau(x; \mathbf{0}) = c_0(x), \quad \partial_k^\hbar \tau(x; \mathbf{t}) \Big|_{\mathbf{t}=\mathbf{0}} = \frac{k}{\hbar} c_k(x), \quad k \geq 1$$

and $c_1 = \partial_x c_0 - c_0 \partial_x \log f$.

$\hbar$ - KP hierarchy

For many applications in physics and mathematics one needs to deal with logarithm of the tau-function rather than with the tau-function itself. Let us put

$$F(x; \mathbf{t}) = \hbar^2 \log \tau(x; \mathbf{t}). \quad (8)$$

Then the Hirota equations on $\tau(x; \mathbf{t})$ go to $\hbar$ - KP hierarchy on $F(x; \mathbf{t})$. That is

$$e^{\Delta(z_1)\Delta(z_2)F} = 1 - \frac{\Delta(z_1)\partial_x F - \Delta(z_2)\partial_x F}{z_1 - z_2}, \quad (9)$$

where

$$\Delta(z) = \frac{e^{\hbar D(z)} - 1}{\hbar}. \quad (10)$$

For $\hbar = 1$ this is the ordinary KP hierarchy.

For $\hbar = 0$ this is the dispersionless KP hierarchy.

Our goal is a formula, expressing any solution of $\hbar$ - KP hierarchy by *Cauchy-like data* $f_k(x) = \partial_k^\hbar F(x; t_1, t_2, \dots) \Big|_{\mathbf{t}=0}$.

The $\hbar$ -KP hierarchy in terms of $\partial_k^{\hbar}$

Define some combinatorial constants $\tilde{P}_{ij}(s_1, \dots, s_m)$ as the number of sequences of positive integers $(i_1, \dots, i_m)$ and $(j_1, \dots, j_m)$ such that $i_1 + \dots + i_m = i$, $j_1 + \dots + j_m = j$ and $s_k = i_k + j_k - 1$. Put

$$P_{ij}(s_1, \dots, s_m) = \frac{(-1)^{m+1} ij}{m s_1 \dots s_m} \tilde{P}_{ij}(s_1, \dots, s_m).$$

Theorem

The $\hbar$ -KP hierarchy is equivalent to the system of equations

$$\partial_i^{\hbar} \partial_j^{\hbar} F = \sum_{m \geq 1} \sum_{\substack{s_1, \dots, s_m \geq 1 \\ s_1 + \dots + s_m = i+j-m}} P_{ij}(s_1, \dots, s_m) \partial \partial_{s_1}^{\hbar} F \dots \partial \partial_{s_m}^{\hbar} F, \quad (11)$$

for the function $F = F(x; \mathbf{t})$, where $\partial = \partial_1$.

Combinatorial constants

Let $K_l(l^1, \dots, l^r)$ be the number of partitions of a set of l elements into ordered groups of $l^1, \dots, l^r > 0$ elements.

Define the constants $P_{i_1 \dots i_k}^{\hbar} \left(\begin{smallmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{smallmatrix} \right)$ from integer positive $m, \{i_r\}, \{s_r\}, \{l_r\}$ by the following recurrence relations:

$$1) \quad P_{i_1, i_2}^{\hbar} \left(\begin{smallmatrix} s_1 \dots s_m \\ 1 \dots 1 \end{smallmatrix} \right) = P_{i_1 i_2}(s_1, \dots, s_m) \text{ and } P_{i_1, i_2}^{\hbar} \left(\begin{smallmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{smallmatrix} \right) = 0, \\ \text{if } \prod_{j=1}^m l_j > 1.$$

$$2) \quad P_{i_1 \dots i_r}^{\hbar} \left(\begin{smallmatrix} x_1 \dots x_v \\ y_1 \dots y_v \end{smallmatrix} \right) = \sum P_{i_1 \dots i_{r-1}}^{\hbar} \left(\begin{smallmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{smallmatrix} \right) \frac{\hbar^{\nu(k_1, \dots, k_m)-1} i_r}{[k_1 \dots k_m]} \\ \times K_{l_1}(l_1^1, \dots, l_1^{n_1}) P_{s_1 k_1}(s_1^1 \dots s_{n_1}^1) \dots K_{l_m}(l_m^1, \dots, l_m^{n_m}) P_{s_m k_m}(s_1^m \dots s_{n_m}^m), \\ \text{where } \nu(k_1, \dots, k_n) \text{ is the number of positive numbers between } k_i \text{ and} \\ [k_1, \dots, k_n] = \prod_{i=1}^n \max\{k_i, 1\}.$$

The summation is carried over all sets of integer numbers such that

$$(x_1 \dots x_v) = (s_1^1, \dots, s_{n_1}^1, s_1^2, \dots, s_{n_2}^2, \dots, s_1^m, \dots, s_{n_m}^m), \quad s_i = \sum_{j=1}^{n_i} s_j^i,$$

$$(y_1 \dots y_v) = (l_1^1 + 1, \dots, l_{n_1}^1 + 1, l_1^2 + 1, \dots, l_{n_2}^2 + 1, \dots, l_1^m + 1, \dots, l_{n_m}^m + 1),$$

$$l_i = \sum_{j=1}^{n_i} l_j^i, \quad \sum_{i=1}^m (s_i + l_i) = \sum_{j=1}^{r-1} i_j, \quad \sum_{i=1}^m k_i = i_r, \quad \sum_{i=1}^{n_j} s_i^j = k_j + s_j.$$

Theorem

The $\hbar$ -KP hierarchy is equivalent to the system of equations for $r \geq 2$

$$\partial_{i_1}^{\hbar} \dots \partial_{i_r}^{\hbar} F = \sum_{m \geq 1} \sum_{\substack{s_1 + l_1 + \dots + s_m + l_m \\ = i_1 + \dots + i_r \\ 1 \leq s_i; 1 \leq l_i \leq r-1}} P_{i_1 \dots i_r}^{\hbar} \left(\begin{matrix} s_1 \dots s_m \\ l_1 \dots l_m \end{matrix} \right) \partial^{i_1} \partial_{s_1}^{\hbar} F \dots \partial^{i_m} \partial_{s_m}^{\hbar} F \quad (12)$$

Operators ∂_λ^h and corresponding variables t_λ^h

A Young diagram $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$ with rows of lengths $\lambda_1, \lambda_2, \dots, \lambda_\ell$ is described by $r_i = \text{card}\{j | \lambda_j = i\}$. Denote by $\rho(\lambda) = \lambda_1 \lambda_2 \dots \lambda_\ell$ and $\sigma(\lambda) = \prod_{n \geq 1} r_n!$. We put also $\partial_\lambda^h = \partial_{\lambda_1}^h \partial_{\lambda_2}^h, \dots, \partial_{\lambda_\ell}^h$.

Let us consider the basis

$$m_\lambda(x_1, x_2, \dots, x_n) = \frac{1}{(n - \ell(\lambda))! \sigma(\lambda)} \sum_{s \in S_n} x_1^{\lambda_{s(1)}} x_2^{\lambda_{s(2)}} \dots x_n^{\lambda_{s(n)}}$$

in the space of symmetrical polynomials from $x_1, x_2, \dots$.

These polynomials are linear combinations from symmetrical polynomials

$t_k = \frac{1}{k} \sum_i x_i^k$. This gives new polynomials $m_\lambda(\mathbf{t})$ from $\mathbf{t} = (t_1, t_2, \dots)$.

The first few functions m_λ are:

$$m_{(1)}(\mathbf{t}) = t_1,$$

$$m_{(2)}(\mathbf{t}) = 2t_2, \quad m_{(1^2)}(\mathbf{t}) = \frac{1}{2} t_1^2 - t_2,$$

$$m_{(3)}(\mathbf{t}) = 3t_3, \quad m_{(12)}(\mathbf{t}) = 2t_2 t_1 - 3t_3, \quad m_{(1^3)}(\mathbf{t}) = \frac{1}{6} t_1^3 - t_2 t_1 + t_3. \quad (13)$$

Let us put

$$t_{\lambda}^{\hbar} := \frac{\sigma(\lambda)}{\rho(\lambda)} \hbar^{\ell(\lambda)} m_{\lambda}(\mathbf{t}/\hbar). \quad (14)$$

The first few $t_{\lambda}^{\hbar}$ for $(\lambda) = (1^{r_1} 2^{r_2}, \dots)$ are:

$$\begin{aligned} t_{(1)}^{\hbar} &= t_1, \\ t_{(2)}^{\hbar} &= t_2, \quad t_{(1^2)}^{\hbar} = t_1^2 - 2\hbar t_2, \\ t_{(3)}^{\hbar} &= t_3, \quad t_{(21)}^{\hbar} = t_2 t_1 - \frac{3}{2}\hbar t_3, \quad t_{(1^3)}^{\hbar} = t_1^3 - 6\hbar t_2 t_1 + 6\hbar^2 t_3. \end{aligned} \quad (15)$$

Theorem

Any formal series $F(\mathbf{t}) = F(t_1, t_2, \dots)$ has a representation in form of formal series

$$F(\mathbf{t}) = \sum_{\lambda} \partial_{\lambda}^{\hbar} F(\mathbf{t}') \Big|_{\mathbf{t}'=0} \frac{t_{\lambda}^{\hbar}}{\sigma(\lambda)}, \quad (16)$$

Construction of formal solutions

Consider now any family of infinitely differentiable functions

$$f_{[k]}^{\hbar}(x) = f_k^{\hbar}(x) \quad (k = 1, 2, \dots).$$

For other Young diagrams λ we put

$$f_{\lambda}^{\hbar}(x) = \sum_{m \geq 1} \sum_{\substack{s_1 + l_1 + \dots + s_m + l_m = |\lambda| \\ 1 \leq s_i; 1 \leq l_i \leq \ell(\lambda) - 1}} P_{\lambda}^{\hbar} \left(\begin{matrix} s_1 \dots s_m \\ l_1 \dots l_m \end{matrix} \right) \partial^{l_1} f_{s_1}(x) \dots \partial^{l_m} f_{s_m}(x), \quad (17)$$

where

$$P_{\lambda}^{\hbar} \left(\begin{matrix} s_1 \dots s_m \\ l_1 \dots l_m \end{matrix} \right) = P_{\lambda_1 \dots \lambda_r}^{\hbar} \left(\begin{matrix} s_1 \dots s_m \\ l_1 \dots l_m \end{matrix} \right)$$

for $\lambda = [\lambda_1, \dots, \lambda_{\ell}]$.

Theorem

For any $\hbar$ and any family of smooth or formal functions

$$\mathbf{f} = \{f_0(x), f_1(x), f_2(x), \dots\}$$

there exists a unique solution $F(x; \mathbf{t})$ of the $\hbar$ -KP hierarchy such that

$$F(x; \mathbf{0}) = f_0(x) \quad \text{and} \quad \partial_k^{\hbar} F(x; t_1, t_2, \dots) \Big|_{\mathbf{t}=\mathbf{0}} = f_k(x).$$

This solution has the form

$$F(x; \mathbf{t}) = f_0(x) + \sum_{|\lambda| \geq 1} \frac{f_{\lambda}^{\hbar}(x)}{\sigma(\lambda)} t_{\lambda}^{\hbar}. \quad (18)$$