Relative Differential Invarinats

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Differential Invariants: Setup

$\mathcal{G} \subset \mathrm{Diff}_{\mathsf{loc}}(M)$ a Lie pseudogroup. Acts on a manifold M and induces an action on n-submanifolds $N \subset M$. Usually identified via Lie equation as $\mathcal{G}_\infty \subset J^\infty(M, n)$.

$\mathbb{E}\mathrm{x}$: (1) Lie group rep; (2) Symmetry of PDE; (3) Gauge action.

Equivalence problem: distinguish germs/jets of submanifolds wrt $\mathcal{G}$.
Partial cases: normal forms, shapes, image recognition, signatures.

Functions constant on orbits – differential invariants. Due to proj limit $J^\infty = \varprojlim J^k$ every differential invariant has a (min) order k.

What abt structure of the space of differential invariants $\mathcal{I} = \cup \mathcal{I}_k$?
Classically: sheaf of rings. More recently: algebra or field.

Rationality property for Differential Invariants

Rational differential invariants are natural in the classification problems and all known examples are such, even though one can encounter many classical expressions with roots.

Example

The proper motion group $E(2)_+ = SO(2) \ltimes \mathbb{R}^2$ acts on the curves in Euclidean $\mathbb{R}^2(x, y)$. The classical curvature

$$K = \frac{y''(x)}{\sqrt{(1 + y'(x)^2)^3}}$$

is not an invariant of $E(2)_+$ (reflection $(x, y) \mapsto (-x, -y)$ preserves the circle $x^2 + y^2 = R^2$ but transforms $K \mapsto -K$; $K = \pm R^{-1}$), but its square K^2 is a bona fide rational differential invariant. Notice however that the Lie group $E(2)_+$ is connected, and that K is invariant under the action of its Lie algebra.

Structure theorem: Global Lie-Tresse

We restrict to the algebra of invariant functions $\mathfrak{A}^l = \cup_k \mathfrak{A}^l_k$ that are smooth by the base variables, rational by the fibers of $\pi_l : J^l \to M$ and polynomial by higher jets of order $k > l$.

Theorem (BK, Valentin Lychagin [Selecta Math 2016])

Consider an algebraic action of a pseudogroup $\mathcal{G}$ on a formally integrable irreducible differential equation $\mathcal{E}$ over M such that $\mathcal{G}$ acts transitively on M. Then $\exists\, l \in \mathbb{N}$ and a Zariski closed invariant proper subset $S_l \subset \mathcal{E}^l$ s.t. the action is regular in the complement, i.e. $\forall k$ there exists rational geometric quotient $\left(\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l)\right)/G^k$.

There exists a finite number of functions $I_1, \dots, I_t \in \mathfrak{A}^l$ and a finite number of rational invariant derivations $\nabla_1, \dots, \nabla_s : \mathfrak{A}^l \to \mathfrak{A}^l$ such that any function from $\mathfrak{A}^l$ is a polynomial of differential invariants $\nabla_J I_i$, where $\nabla_J = \nabla_{j_1} \cdots \nabla_{j_r}$ for some multi-indices $J = (j_1, \dots, j_r)$, with coefficients being rational functions of I_i.

From absolute to relative invariants

Note that $\mathbb{P}^n = \mathbb{R}^{n+1}/\mathbb{R}_\times$ possesses no (global) polynomial functions, but it has many rational functions $R([z]) = P(z)/Q(z)$, where $[z] = [z_0 : z_1 : \cdots : z_n]$ and $P(z), Q(z)$ are polynomials of $z = (z_0, z_1, \ldots, z_n)$ of the same degreee d. In other words, P, Q are sections of the line bundle $\mathcal{L} = \mathcal{O}(d)$ over $\mathbb{P}^n$. Note that though $SL(n)$ acts on $\mathbb{P}^n$, $GL(n)$ acts on the total space of $\mathcal{L}$.

NB: Modular forms can be approached from this geometric perspective, as sections of line bundles over moduli space of elliptic curves. Automorphic forms extend this to other groups/varieties.

Similarly, since the differential invariants of $\mathcal{G}$ on $\mathcal{E}$ are shown (under the assumption of algebraicity of the action and transitiveity on the base) to be rational functions in jet-variables, it is worth studying the corresponding polynomial relative invariants.

Example: Plane projective geometry

Prolong the action of $G = PSL(3)$ on $\mathbb{R}^2(x,u) = J^0(\mathbb{R},\mathbb{R})$ to $J^k = \mathbb{R}^{k+2}(x,u_0,u_1,\dots,u_k)$. The corresponding Lie algebra is

$$\mathfrak{g} = \langle \partial_x, \partial_u, x\partial_x, x\partial_u, u\partial_x, u\partial_u, x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u \rangle.$$

One can check that the following are relative invariants:

$$J_2 = u_2, \qquad J_5 = 9u_2^2u_5 - 45u_2u_3u_4 + 40u_3^3$$

$$J_7 = 162u_2^6u_5u_7 - 189u_2^6u_6^2 - 810u_2^5u_3u_4u_7 + 1134u_2^5u_3u_5u_6$$
$$+1890u_2^5u_4^2u_6 - 2835u_2^5u_4u_5^2 + 720u_2^4u_3^3u_7 - 3150u_2^4u_3^2u_4u_6 - 756u_2^4u_3^2u_5^2$$
$$+ 13230u_2^4u_3u_4^2u_5 - 4725u_2^4u_4^4 - 12600u_2^3u_3^3u_4u_5 - 7875u_2^3u_3^2u_4^3$$
$$+ 6720u_2^2u_3^5u_5 + 31500u_2^2u_3^4u_4^2 - 33600u_2u_3^6u_4 + 11200u_3^8$$

and the weights of J_5 and J_7 are proportional. The weight lattice $\mathcal{W}$ is 2-dim: generated by $\lambda_1(X) = \mathrm{div}_\omega(X)$, $\lambda_2(X) = \mathrm{div}_\Omega(X^{(1)})$ for $\omega = dx \wedge du$, $\Omega = \alpha \wedge d\alpha$, $\alpha = du - u_1dx$.

Absolute inv: $\mathfrak{A}^5 = \langle I_7 = J_7^3/I_5^8, \nabla = \frac{J_2J_7}{J_5^3}D_x \rangle$.

Classification of relative invariants

Let $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ be the Lie algebra sheaf corresponding to the pseudogroup. Then P is an absolute differential invariant if $L_X P = 0$ (assume $\mathcal{G}$ connected for simplicity), and R is a relative differential invariant if $L_X R = \lambda(X) R \; \forall X \in \mathfrak{g}$, where we tacitly omited the prolongation sign and $\lambda \in \mathfrak{g}^* \otimes \mathfrak{R}(\mathcal{E})$.

Indeed, the consistency $L_X L_Y R - L_Y L_X R = L_{[X,Y]} R$ implies the 1-cocycle condition for λ. Moreover rescaling $R \mapsto fR$ results in the shift $\lambda \mapsto \lambda + d \log f$ by coboundary. Thus λ defines an element in the Gelfand-Fuks cohomology $H^1(\mathfrak{g}, \mathfrak{R}(\mathcal{E}))$; note that both the Lie algebra and module are infinite-dimensional. Having computed this one checks realizablity of cocycles: which λ possess R?

Two conditions arise due to local property of the pseudogroup:
(1) The cocycles in $Z^1(\mathfrak{g}, \mathfrak{R}(\mathcal{E}))$ should be local (finite jet);
(2) The rational behavior should not confront $\log$-modulus.

Recollection: Homogeneous vector bundles

If $M = G/H$ is a homogeneous space and V a representation of H then $E = G \times_H V$ is a bundle over M with typical fiber V. It is called a homogeneous VB. In particular, homogeneous line bundles are classified by 1D rep of H.

For generalized flag varieities G/P (G semi-simple, P parabolic) the line bundles are parametrized by the lattice $\mathbb{Z}^{|\mathfrak{z}(\mathfrak{p})|}$. Refall that semi-simple Lie algebras have $\mathbb{Z}$-gradations in $1:1$ with parabolics:

$$\mathfrak{g} = \mathfrak{g}_{-\nu} \oplus \cdots \oplus \underbrace{\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\nu}_{\mathfrak{p}}$$

Then $\mathfrak{g}_0$ is reductive with $\mathfrak{z}(\mathfrak{p}) = \mathfrak{z}(\mathfrak{g}_0)$.

Moreover for $M = G/P$ the model tangent has filtration associated to

$$\mathfrak{m} = T_o M \simeq \mathfrak{g}_{-\nu} \oplus \cdots \oplus \mathfrak{g}_{-1}.$$

Cartan and parabolic geometries

A Cartan geometry of type (G, H) is a H-principal bundle $H \to \mathcal{P} \to M$ and an affine connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, which is:
(1) H-equivariant, (2) iso $T\mathcal{P} \simeq \mathfrak{g}$,
(3) reproduces the fundamental fields from $\mathfrak{h} = \mathrm{Lie}(H)$.

A parabolic geometry is a Cartan geometry of type (G, P), where G is semi-simple and P parabolic. This class includes conformal, projective, Levi-nondegenerate CR, Legendrian contact, special vector distributions and other geometries.

Theorem

For parabolic geometries of type (G, P) the weight lattice $\mathcal{W}$ of the relative invariants is $\mathbb{R}^{|\mathfrak{z}(\mathfrak{p})|}$.

The weight lattice $\mathcal{W} = \{\lambda\}$ consisting of realizable 1-cocycles can be identified with the quotient space $\mathfrak{A}^l_{\mathsf{rel}}/\mathfrak{A}^l$.

Example: second order ODEs

Consider second order ODEs $y'' = f(x, y, y')$ mod point transformations $\psi \in \mathcal{G} = \mathrm{Diff}_{\mathsf{loc}}(\mathbb{R}^2(x, y))$; let $p = y'$.

Relative invariants were classified by A.Tresse [Acta Math 1896].
The absolute invariants were derived by BK [Abel Symp 2008].
The lattice $\mathcal{W}$ is generated by the following 1-cocycles:

- Standard divergence: $X = a\partial_x + b\partial_y \mapsto a_x + b_y = \mathrm{div}_\omega(X)$, $\omega = dx \wedge dy$ for $J^0 = \mathbb{R}^2(x, y)$; and for $J^1 = \mathbb{R}^3(x, y, p)$
- Contact divergence: $X^{(1)} = X + (b_x - a_x p + b_y p - a_y p^2)\partial_p \mapsto (b - ap)_y = \frac{1}{2}\,\mathrm{div}_\Omega(X^{(1)})$, where $\Omega = \alpha \wedge d\alpha = dx \wedge dy \wedge dp$.

There are two basic relative invariants: $I = f_{pppp}$ and

$$H = f^2 f_{pppp} + 2pf f_{yppp} + p^2 f_{yypp} - p f_p f_{ypp} + p f_y f_{ppp} - 3 f f_{ypp} + 2 f f_{xppp} - 4p f_{yyp} + 2p f_{xypp} + 4 f_p f_{yp} - f_p f_{xpp} + f_x f_{ppp} - 3 f_y f_{pp} + 6 f_{yy} - 4 f_{xyp} + f_{xxpp}$$

More examples: projective connections, conformal structures, etc.

Example: (2,3,5) distributions

Consider a non-integrable distribution $\Delta^2 \subset TM^5$ with growth vector $(2,3,5)$. Encoded by Monge underdetermined ODEs $y' = f(x,y,z,z',z'')$. Maximal symmetry algebra is the exceptional Lie group G_2, achieved when the Cartan quartic invariant

$$\Psi(\Delta) = A = a_4p^4 + 4a_3p^3q + 6a_2p^2q^2 + 4a_1pq^3 + a_0q^4$$

vanishes. Here $\Delta \simeq \mathbb{R}^2(p,q)$ is a natural representation of $GL(2)$. There are two relative invariants of $GL(2)$ on quartics $S^4\Delta^*$

$$r_2(A) = a_0a_4 - 4a_1a_3 + 3a_2^2,\ r_3(A) = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3$$

that produce relative differential invariants of weights 2 and 3:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow r_2(gA) = (\det g)^4 r_2(A),\ r_3(gA) = (\det g)^6 r_3(A).$$

Thus $I(\Delta) = r_3^2(A)/r_2^3(A)$ is an absolute differential invariant of the generic rank 2 distribution... $\rightsquigarrow$ algebra $\mathfrak{A}^6$.

Applications to integrability

(1) Darboux integrability via Laplace invariants;

$$u_{xy} = f(x, y, u, u_x, u_y) \rightsquigarrow \text{ general hyperbolic 2nd ord PDE.}$$

(2) Liouville integrability of geodesic flows;

$$g = g_{ij}(x)dx^i dx^j \Leftrightarrow H = \tfrac{1}{2}g^{ij}(x)p_i p_j : \quad \{H, F\} = 0.$$

(3) Integrability via geometry: EW, SD etc.

Hirota type eqn $H(u_{ij}) = 0,\ 1 \leq i \leq j \leq d\ (d = 2, 3, 4).$