

String amplitudes from genus zero to genus one

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Joint work with Don Zagier (arXiv:1906.12339 [math.NT])

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String amplitudes

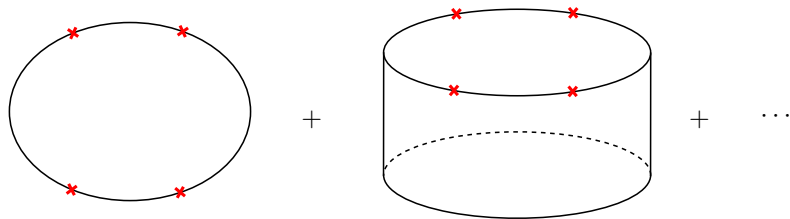


Figure: Four open strings

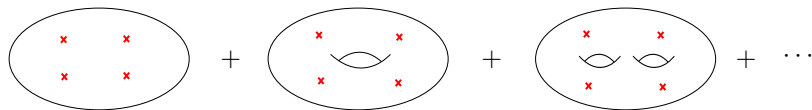


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Four-point amplitudes at genus zero

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Open strings: the Veneziano amplitude

$$A_{0,4}^{(\text{op})}(s, t) = \int_{[0,1]} x^{s-1} (1-x)^{t-1} dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

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Closed strings: the Virasoro-Shapiro amplitude

$$A_{0,4}^{(\text{cl})}(s, t) = \int_{\mathbb{P}_{\mathbb{C}}^1} |z|^{2s-2} |1-z|^{2t-2} \frac{dz d\bar{z}}{(-2\pi i)} = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s-t)}{\Gamma(1-s)\Gamma(1-t)\Gamma(s+t)}.$$

The single-valued projection

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Laurent expansion of $A_{0,4}^{(\text{op})}(s, t)$

$$A_{0,4}^{(\text{op})}(s, t) = \frac{s+t}{st} \exp \left(\sum_{n \geq 2} \frac{(-1)^n \zeta(n)}{n} (s^n + t^n - (s+t)^n) \right).$$

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Define $\text{sv}(\zeta(2k)) = 0$ and $\text{sv}(\zeta(2k+1)) = 2\zeta(2k+1)$

Then $\text{sv}(A_{0,4}^{(\text{op})}(s, t)) = A_{0,4}^{(\text{cl})}(s, t)$.

Genus one, two points, **closed** strings

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The Green's function on the torus

Let $\tau \in \mathbb{H}$, $\mathcal{E}_\tau := \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$, $z \in \mathcal{E}_\tau$. Define

$$G(z, \tau) := -\log \left| \frac{\theta_1(z, \tau)}{\eta(\tau)} \right|^2 + \frac{2\pi \operatorname{Im}(z)^2}{\operatorname{Im}(\tau)}.$$

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The configuration space amplitude

Consider the measure $d\mu(z) = \frac{i dz d\bar{z}}{2 \operatorname{Im}(\tau)}$ on $\mathcal{E}_\tau$ and define

$$A_{1,2}^{(\text{cl})}(s, \tau) := \int_{\mathcal{E}_\tau} e^{s G(z, \tau)} d\mu(z) = \sum_{l \geq 0} \frac{s^l}{l!} \int_{\mathcal{E}_\tau} G(z, \tau)^l d\mu(z).$$

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Definition (D'Hoker, Green, Gurdogan, Vanhove)

We call $D_l(\tau) := \int_{\mathcal{E}_\tau} G(z, \tau)^l d\mu(z)$ (2-point) **modular graph functions**.

Two-point modular graph functions

Two-point modular graph functions

Lattice sum representation

Since $G(u\tau + v, \tau) = \frac{\text{Im}(\tau)}{\pi} \sum_{m,n} \frac{\exp(2\pi i(nu - mv))}{|m\tau + n|^2}$ then

$$D_l(\tau) = \left(\frac{\text{Im}(\tau)}{\pi} \right)^l \sum_{\substack{m_1, n_1, \dots, m_l, n_l \\ \sum m_i = \sum n_i = 0}} \frac{1}{|m_1\tau + n_1|^2 \cdots |m_l\tau + n_l|^2}.$$

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In particular $D_l(\gamma\tau) = D_l(\tau)$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

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In particular $D_l(\gamma\tau) = D_l(\tau)$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

Examples

$$D_1(\tau) = 0$$

$$D_2(\tau) = E(2, \tau)$$

$$D_3(\tau) = E(3, \tau) + \zeta(3)$$

$$D_4(\tau) = \frac{Y^4}{945} + \frac{2\zeta(3)Y}{3} + \frac{10\zeta(5)}{Y} - \frac{3\zeta(3)^2}{Y^2} + \frac{9\zeta(7)}{4Y^3} + O(e^{-Y}), \quad Y := 2\pi\text{Im}(\tau)$$

The leading term as $\tau \rightarrow i\infty$

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Proposition (Green, Russo, Vanhove; Zagier)

$$A_{1,2}^{(\text{cl})}(s, \tau) = e^{sY/6} \sum_{n=0}^{\infty} \left(\frac{Y^n}{(2n+1)!!} - \sum_{k=1}^{n-1} \frac{(2k-3)!! \gamma_{n,k}}{(-Y)^k} \right) \left(\frac{-s}{2} \right)^n \\ + O(e^{-Y}) \quad (Y := 2\pi \text{Im}(\tau) \rightarrow +\infty),$$

where the coefficients $\gamma_{n,k}$ are given by the special MZVs

$$\gamma_{n,k} := \sum_{r=0}^{n-k-1} (-2)^{r+2} \binom{n-r+k-3}{2k-2} Z(k+n-r, r) \quad (0 < k < n),$$

$$Z(m, r) := \sum_{\substack{\mathbf{r} \in \{1,2\}^j, j \geq 0 \\ r_1 + \dots + r_j = r}} 2^{\#\{i: r_i=1\}} \zeta(\mathbf{r}, m) \quad (m \geq 2, r \geq 0).$$

The numbers $\gamma_{n,k}$

$$\gamma_{2,1} = 4\zeta(3)$$

$$\gamma_{3,1} = 0$$

$$\gamma_{4,1} = 12\zeta(5)$$

$$\gamma_{5,1} = -8\zeta(3)^2$$

$$\gamma_{6,1} = 36\zeta(7)$$

$$\gamma_{3,2} = 4\zeta(5)$$

$$\gamma_{4,2} = 8\zeta(3)^2$$

$$\gamma_{5,2} = 24\zeta(7)$$

$$\gamma_{6,2} = 32\zeta(5)\zeta(3)$$

$$\gamma_{7,2} = \frac{292}{3}\zeta(9) - \frac{64}{3}\zeta(3)^2$$

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Define the generating function

$$\mathcal{G}(X, Y) := \frac{1}{X(X+Y)(Y-X)} + \sum_{n>k>0} \gamma_{n,k} X^{n-k-1} Y^{2k-2}$$

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Theorem (Zagier, Z.)

$$\mathcal{G}(X, Y) = \frac{2 A_{0,4}^{(\text{cl})}(2X, -X - Y)}{(Y - X)^2}.$$

Genus one, two points, **open** strings

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Restrict $G(z, \tau)/2$ to $\tau \in i\mathbb{R}^+$ and “B-cycle” $z \in \mathbb{R}\tau/\mathbb{Z}\tau$ and get

$$P(z, \tau) = -\log \left| \frac{\theta_1(z, \tau)}{\eta(\tau)} \right| - \frac{i\pi}{\tau} \left(z^2 + \frac{1}{6} \right).$$

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Define the (cylinder, 2-point) configuration space amplitude

$$A_{1,2}^{(\text{op})}(s, \tau) := \int_{\mathbb{R}\tau/\mathbb{Z}\tau} e^{s P(z, \tau)} \frac{dz}{\tau} = \sum_{l \geq 0} \frac{s^l}{l!} \int_{\mathbb{R}\tau/\mathbb{Z}\tau} P(z, \tau)^l \frac{dz}{\tau}.$$

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Unique holomorphic extensions of $B_l(\tau) := \int_{\mathbb{R}\tau/\mathbb{Z}\tau} P_B(z, \tau)^l dz/\tau$ to $\mathbb{H}$ are called (2-point) **holomorphic graph functions**.

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Examples

$$B_1(\tau) = 0, \quad B_2(\tau) \sim \tilde{G}_4(\tau), \quad B_3(\tau) \sim \tilde{G}_6(\tau).$$

The leading term as $\tau \rightarrow i\infty$

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Proposition (Zagier, Z.)

$$A_{1,2}^{(\text{op})} = e^{sT/6 - \zeta(2)s/T} \sum_{n \geq 0} \left(\frac{T^n}{(2n+1)!!} - \sum_{k=1}^n \frac{(2k-3)!! \eta_{n,k}}{(-T)^k} \right) \left(-\frac{s}{2} \right)^n + O(e^{-T}) \quad (T := \pi\tau/i \rightarrow +\infty),$$

where the coefficients $\eta_{n,k}$ are given for $0 < k \leq n$ by the special MZVs

$$\eta_{n,k} := \sum_{r=-1}^{n-k-1} (-2)^{r+2} \binom{k+n-r-3}{2k-2} \zeta(\underbrace{1, \dots, 1}_{r+1}, k+n-r-1).$$

The numbers $\eta_{n,k}$

$$\eta_{1,1} = -\zeta(2)$$

$$\eta_{2,1} = 2\zeta(3)$$

$$\eta_{3,1} = -9\zeta(4)$$

$$\eta_{4,1} = 6\zeta(5) + 4\zeta(3)\zeta(2)$$

$$\eta_{5,1} = -2\zeta(3)^2 - \frac{61}{2}\zeta(6)$$

$$\eta_{2,2} = -2\zeta(4)$$

$$\eta_{3,2} = 2\zeta(5) - 4\zeta(3)\zeta(2)$$

$$\eta_{4,2} = 2\zeta(3)^2 - \frac{29}{2}\zeta(6)$$

$$\eta_{5,2} = 12\zeta(7) - 14\zeta(3)\zeta(4) - 8\zeta(5)\zeta(2)$$

$$\eta_{6,2} = 8\zeta(3)\zeta(5) + 8\zeta(3)^2\zeta(2) - \frac{703}{12}\zeta(8)$$

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Fact:
$$\sum_{k,r \geq 1} \zeta(\underbrace{1, \dots, 1}_{r-1}, k+1) s^k t^r = 1 - \frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(1-s-t)}.$$

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Corollary: All numbers $\eta_{n,k}$ are polynomials in single zeta values with rational coefficients.

Genus one vs genus zero

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Define the generating function

$$\mathcal{E}(X, Y) := \frac{1}{X(X+Y)(Y-X)} + \sum_{n \geq k > 0} \eta_{n,k} X^{n-k-1} Y^{2k-2}$$

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Proposition (Zagier, Z.)

The numbers $\eta_{n,k}$ are polynomials in single zeta values with rational coefficients, explicitly given in terms of the coefficients of $A_{0,4}^{(\text{op})}$ by

$$\mathcal{E}(X, Y) = \left(\frac{\sin(\pi(X+Y))}{\sin(\pi(Y-X))} - 1 \right) \frac{A_{0,4}^{(\text{op})}(2X, -X-Y)}{X(Y-X)}.$$

Single-valued projection at genus 1

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Compare:

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$$\gamma_{1,1} = 0$$

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$$\gamma_{7,3} = 80\zeta(7)\zeta(3) + 40\zeta(5)^2$$

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$$\gamma_{7,3} = 80\zeta(7)\zeta(3) + 40\zeta(5)^2$$

Conjecture (Broedel, Schlotterer, Z. - 2018): $\text{sv}(\eta_{n,k}) = \gamma_{n,k}$

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Closed strings

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Single-valued projection at genus 1

$$\zeta(2) \rightarrow 0, \quad \eta_{n,k} \rightarrow \gamma_{n,k}, \quad T \rightarrow Y \quad (\log q \rightarrow \log |q|^2)$$

Open strings

$$A_{1,2}^{(\text{op})} = e^{sT/6 - \zeta(2)s/T} \sum_{n \geq 0} \left(\frac{T^n}{(2n+1)!!} - \sum_{k=1}^n \frac{(2k-3)!! \eta_{n,k}}{(-T)^k} \right) \left(-\frac{s}{2} \right)^n \\ + O(e^{-T}) \quad (T := \pi\tau/i \rightarrow +\infty).$$

Closed strings

$$A_{1,2}^{(\text{cl})}(s, \tau) = e^{sY/6} \sum_{n=0}^{\infty} \left(\frac{Y^n}{(2n+1)!!} - \sum_{k=1}^{n-1} \frac{(2k-3)!! \gamma_{n,k}}{(-Y)^k} \right) \left(\frac{-s}{2} \right)^n \\ + O(e^{-Y}) \quad (Y := 2\pi \text{Im}(\tau) \rightarrow +\infty),$$

Single-valued projection at genus 1

Recall:

- $\mathcal{G}(X, Y) = 2 A_{0,4}^{(\text{cl})}(2X, -X - Y)/(Y - X)^2$

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- $\mathcal{G}(X, Y) = 2 A_{0,4}^{(\text{cl})}(2X, -X - Y)/(Y - X)^2$
- $\mathcal{E}(X, Y) = \left(\frac{\sin(\pi(X+Y))}{\sin(\pi(Y-X))} - 1 \right) A_{0,4}^{(\text{op})}(2X, -X - Y)/X(Y - X)$

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Theorem (Zagier, Z.)

$\text{sv}(\mathcal{E}(X, Y)) = \mathcal{G}(X, Y)$ and therefore $\text{sv}(\eta_{n,k}) = \gamma_{n,k}$

Single-valued projection at genus 1

Recall:

- $\mathcal{G}(X, Y) = 2 A_{0,4}^{(\text{cl})}(2X, -X - Y)/(Y - X)^2$
- $\mathcal{E}(X, Y) = \left(\frac{\sin\left(\frac{\pi(X+Y)}{\pi(Y-X)}\right)}{\sin\left(\frac{\pi(Y-X)}{\pi(Y-X)}\right)} - 1 \right) A_{0,4}^{(\text{op})}(2X, -X - Y)/X(Y - X)$
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Theorem (Zagier, Z.)

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Proof: $\text{sv}\left(\frac{\sin\left(\frac{\pi(X+Y)}{\pi(Y-X)}\right)}{\sin\left(\frac{\pi(Y-X)}{\pi(Y-X)}\right)} - 1\right) = 2X/(Y - X).$

Summary

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$$\begin{array}{ccc} A_{0,4}^{(\text{op})}(s, t) & \xrightarrow{\text{sv}} & A_{0,4}^{(\text{cl})}(s, t) \\ \tau \rightarrow i\infty \uparrow & & \tau \rightarrow i\infty \uparrow \\ A_{1,2}^{(\text{op})}(s, \tau) & \xrightarrow{\text{sv}} & A_{1,2}^{(\text{cl})}(s, \tau) \end{array}$$

THE END

Thanks for your attention!