# Mirror symmetry for simple elliptic singularities and theta constants

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# Mirror pairs

For every fixed $\sigma \in \mathbb{C}$ assume

- $\tilde{E}_6 : \quad W_\sigma(\mathbf{x}) := x_1^3 + x_2^3 + x_3^3 + \sigma x_1 x_2 x_3,$
- $\tilde{E}_7 : \quad W_\sigma(\mathbf{x}) := x_1^4 + x_2^4 + x_3^2 + \sigma x_1^2 x_2^2,$
- $\tilde{E}_8 : \quad W_\sigma(\mathbf{x}) := x_1^6 + x_2^3 + x_3^2 + \sigma x_1^4 x_2.$

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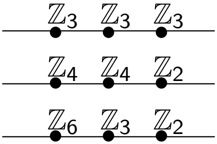
- $\tilde{E}_7$ : $\quad W_\sigma(\mathbf{x}) := x_1^4 + x_2^4 + x_3^2 + \sigma x_1^2 x_2^2$,
- $\tilde{E}_8$ : $\quad W_\sigma(\mathbf{x}) := x_1^6 + x_2^3 + x_3^2 + \sigma x_1^4 x_2$.

Theorem[Satake-Takahashi, Milanov-Ruan-Shen]

Simple elliptic singularity $W_\sigma$ is mirror to orbifold $X$

$$X := \begin{cases} \mathbb{P}^1_{3,3,3}, & \text{for } \tilde{E}_6, \\ \mathbb{P}^1_{4,4,2}, & \text{for } \tilde{E}_7, \\ \mathbb{P}^1_{6,3,2}, & \text{for } \tilde{E}_8. \end{cases}$$




so-called *elliptic orbifolds*.

# Gromov–Witten theory

Let $X$ be a (projective, algebraic) variety $\beta \in H_2(X, \mathbb{Z})$ and $\overline{\mathcal{M}}_{g,n}(X, \beta)$ – the moduli space of stable maps to $X$.

- $\gamma_1, \dots, \gamma_\mu \in H^*(X)$ — the basis of the cohomology ring.

Define:

$$\langle \gamma_{i_1}, \dots, \gamma_{i_n} \rangle_{g,n,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_{i_1}^* \gamma_{i_1} \wedge \dots \wedge ev_{i_n}^* \gamma_{i_n},$$

where $ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ – *evalutation morphisms*.

Definition:

The genus $g$ GW potential $\mathcal{F}_g^X = \mathcal{F}_g^X(t_1, \dots, t_\mu)$ reads.

$$\mathcal{F}_g^X := \sum_{n,\beta} \frac{1}{n!} \langle \gamma_{i_1}, \dots, \gamma_{i_n} \rangle_{g,n,\beta}^X t_{i_1} \cdots t_{i_n}.$$

# Frobenius manifold of elliptic orbifolds

- $\eta = (\eta_{ij})$ is a constant non–degenerate bilinear form

$$\eta_{ij} := \frac{\partial^3 \mathcal{F}_0^X}{\partial t^1 \partial t^p \partial t^q}, \quad 1 \le i, j \le \mu.$$

- for every fixed $1 \le i, j, k, l \le \mu$, $\mathcal{F}_0^X$ satisfies **WDVV eq.**

$$\sum_{p,q} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_i \partial t_j \partial t_p} \eta^{pq} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_q \partial t_k \partial t_l} = \sum_{p,q} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_i \partial t_k \partial t_p} \eta^{pq} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_q \partial t_j \partial t_l}$$

## Proposition

For $X = \mathbb{P}_{a_1,a_2,a_3}$, $\mathcal{F}_0^X$ defines a Frobenius manifold structure $M_X := (\mathbb{C}^{\mu-1} \times \mathbb{H}, \circ, \eta)$ with $\mu := \sum_{k=1}^3 a_k - 1$:

$$\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j} := \sum_{p=1}^{\mu} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_i \partial t_j \partial t_p} \eta^{pk} \frac{\partial}{\partial t_k}, \;\; \eta \left( \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right) := \eta_{ij}$$

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- for every fixed $1 \le i, j, k, l \le \mu$, $\mathcal{F}_0^X$ satisfies **WDVV eq.** = associativity equation

$$\sum_{p,q} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_i \partial t_j \partial t_p} \eta^{pq} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_q \partial t_k \partial t_l} = \sum_{p,q} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_i \partial t_k \partial t_p} \eta^{pq} \frac{\partial^3 \mathcal{F}_0^X}{\partial t_q \partial t_j \partial t_l}$$

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# The unfolding of $W_\sigma$

$$\mathcal{L}_W := \mathbb{C}[x_1, x_2, x_3] \Big/ \left( \frac{\partial W_\sigma}{\partial x_1}, \frac{\partial W_\sigma}{\partial x_2}, \frac{\partial W_\sigma}{\partial x_3} \right),$$

and $\mu := \dim(\mathcal{L}_W) < \infty$.

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and $\mu := \dim(\mathcal{L}_W) < \infty$. Consider the **unfolding** of $W_\sigma(\mathbf{x})$:

$$W(\mathbf{x}, s_1, \dots, s_\mu) := W(\mathbf{x}) + \sum_{k=1}^{\mu} s_k \phi_k(\mathbf{x}),$$

- $\mathbb{C}\langle \phi_1, \dots, \phi_\mu \rangle = \mathcal{L}_W$ and $\phi_1 = 1$,
- $(s_1, \dots, s_\mu) \in \mathcal{S} \subset \mathbb{C}^\mu$ — **the base space of the unfolding**

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Endow $\mathcal{TS}$ with the product $\circ$ having the str.constants $c_{ij}^k(\mathbf{s})$

$$\frac{\partial W}{\partial s_i} \frac{\partial W}{\partial s_j} = c_{ij}^k(\mathbf{s}) \frac{\partial W}{\partial s_k} \quad \mod \left( \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \frac{\partial W}{\partial x_3} \right)$$

# Saito theory

By fixing a volume form $\zeta = \psi(\mathbf{s}) dx_1 \wedge dx_2 \wedge dx_3$ the space $\mathcal{TS}$ is endowed with a pairing $\eta$:

$$\eta\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right) = \mathrm{res}\left[\frac{\frac{\partial W}{\partial s_i}\frac{\partial W}{\partial s_j}}{\frac{\partial W}{\partial x_1}\frac{\partial W}{\partial x_2}\frac{\partial W}{\partial x_3}}\right], \quad W = W(\mathbf{x}, \mathbf{s}).$$

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A volume form $\zeta$, s.t. $\eta$ is flat is called **primitive form** (M. Saito and K.Saito).

Denote

$M_{W,\zeta} := (\mathcal{S}, \circ, \eta)$ the data above with a primitive form $\zeta$.

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Mirror Theorem[Satake-Takahashi, Milanov-Ruan-Shen]:

There is a choice of a primitive form $\zeta = \zeta^{\infty}$, such that $M_{W_\sigma, \zeta^{\infty}}$ is isomorphic to $M_X$ of the respective elliptic orbifold.

# Gromov-Witten theory revisited

Proposition[Satake-Takahashi,B.]

For every elliptic orbifold $X$ there is a homogeneous polynomial $P_X(z_1,\dots,z_k,t^1,\dots,t^\mu)$, s.t.

$$\mathcal{F}_0^X = P_X(z_1,\dots,z_k,t^1,\dots,t^\mu)\mid_{z_\bullet = Z_\bullet(q)}, \quad q = \exp(t^\mu),$$

for the certain $Z_k \in \mathbb{Q}[[q]]$, being Fourier expansions of the certain (quasi)modular forms.

# $g = 0$ Gromov–Witten potential of $\mathbb{P}_{4,4,2}$

$$
\begin{aligned}
\mathcal{F}_0^{\mathbb{P}_{4,4,2}} &= \frac{1}{2} t_1^2 t_9 + t_1 \left( \frac{t_2 t_4}{4} + \frac{t_3^2}{8} + \frac{t_5 t_7}{4} + \frac{t_6^2}{8} + \frac{t_8^2}{4} \right) \\
&+ \frac{x^\infty}{8} \left( t_3 t_2^2 + t_5^2 t_6 \right) + \frac{y^\infty}{8} \left( t_6 t_2^2 + t_3 t_5^2 \right) + \frac{z^\infty}{4} t_2 t_5 t_8 \\
&- w^\infty \left( \frac{t_3^4}{128} + \frac{1}{64} t_6^2 t_3^2 + \frac{1}{32} t_8^2 t_3^2 + \frac{1}{32} t_2 t_4 t_3^2 + \frac{1}{32} t_5 t_7 t_3^2 + \frac{t_6^4}{128} \right. \\
&\left. \cdots + \frac{1}{16} t_2 t_4 t_8^2 + \frac{1}{16} t_5 t_7 t_8^2 + \frac{1}{32} t_5 t_6^2 t_7 + \frac{1}{16} t_2 t_4 t_5 t_7 \right) \\
&+ \frac{x^\infty z^\infty}{16} \left( t_3 t_4 t_5 t_8 + t_2 t_6 t_7 t_8 \right) + \frac{y^\infty z^\infty}{16} \left( t_4 t_5 t_6 t_8 + t_2 t_3 t_7 t_8 \right) + \dots
\end{aligned}
$$

where

$$
x^\infty := \vartheta_3(q^8)^2, \; y^\infty := \vartheta_2(q^8)^2, \; z^\infty := \vartheta_2(q^4)^2,
$$

$$
w^\infty := \frac{1}{3} \left( 4E_2(q^{16}) - 2E_2(q^8) + E_2(q^4) \right), \; q = \exp(t_9).
$$

### Proposition

WDVV equation for $\mathcal{F}_0^{\mathbb{P}_{4,4,2}}$ is equivalent to the following system

$$\frac{\partial}{\partial t}x(t) = x(t)\left(2y(t)^2 - x(t)^2 + w(t)\right),$$
$$\frac{\partial}{\partial t}y(t) = y(t)\left(2x(t)^2 - y(t)^2 + w(t)\right),$$
$$\frac{\partial}{\partial t}w(t) = w(t)^2 - x(t)^4.$$

for which $(x^\infty, y^\infty, z^\infty, w^\infty)$ is a solution.

# Global mirror symmetry

Fix some $W$ defining isolated singularity.

A-side: ?FJRW theory — Frobenius manifold; GW theory — Frobenius manifold

$\zeta^0$ $\zeta^\infty$

B-side: Frobenius manifold of $W, \zeta$

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Fix some $W$ defining isolated singularity.

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?FJRW theory — GW theory

Frobenius manifold — Frobenius manifold

$\zeta^0$ — $\zeta^\infty$

B-side:

Frobenius manifold of $W, \zeta$

Mirror symmetry = classification of **all** Frobenius manifold structures $M_{W,\zeta}$ up to isomorphism.

# $\mathrm{SL}(2,\mathbb{C})$–action on Frobenius manifold potentials

Let $F = F(t_1, \dots, t_\mu)$ be a Frobenius manifold potential and

$$F = \frac{t_1^2 t_\mu}{2} + \frac{t_1}{2} \sum_{i,j=2}^{\mu-1} \eta_{ij} t_i t_j + H(t_2, \dots, t_\mu)$$

for some $H = H(t_2, \dots, t_\mu)$.

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for some $H = H(t_2, \ldots, t_\mu)$. Then for any $A \in \mathrm{SL}(2,\mathbb{C})$

$$F^A := \frac{t_1^2 t_\mu}{2} + \frac{t_1}{2} \sum_{i,j=2}^{\mu-1} \eta_{ij} t_i t_j + \frac{c\left(\sum_{i,j=2}^{\mu-1} \eta_{ij} t_i t_j\right)^2}{8(ct_\mu + d)}$$
$$+ (ct_\mu + d)^2 H\left(\frac{t_2}{ct_\mu + d}, \ldots, \frac{t_{\mu-1}}{ct_\mu + d}, \frac{at_\mu + b}{ct_\mu + d}\right), \; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a Frobenius manifold potential too (cf. Dubrovin, *Geometry of 2D topological field theories*, Appendix B).

# Acting on Frobenius manifold potential of $\mathbb{P}_{4,4,2}$

For the certain polynomial $P_{\mathbb{P}_{4,4,2}} \in \mathbb{Q}[x, y, z, w, t_1, \dots, t_9]$ we had:

$$\mathcal{F}_0^{\mathbb{P}_{4,4,2}} = P_{\mathbb{P}_{4,4,2}} \,|_{x=x^\infty(t_9), y=y^\infty(t_9), z=z^\infty(t_9), w=w^\infty(t_9)} \,.$$

## Proposition

For any $A \in \mathrm{SL}(2,\mathbb{C})$ we have

$$\left(\mathcal{F}_0^{\mathbb{P}_{4,4,2}}\right)^A = P_{\mathbb{P}_{4,4,2}} \,|_{x=x^A(t_9), y=y^A(t_9), z=z^A(t_9), w=w^A(t_9)} \,.$$

$$x^A := \frac{1}{ct_8+d} x^\infty\left(\frac{at_9+d}{ct_9+d}\right), \; y^A := \frac{1}{ct_8+d} y^\infty\left(\frac{at_9+d}{ct_9+d}\right),$$

$$z^A := \frac{1}{ct_8+d} z^\infty\left(\frac{at_9+d}{ct_9+d}\right),$$

$$w^A := \frac{1}{(ct_8+d)^2} w^\infty\left(\frac{at_9+d}{ct_9+d}\right) - \frac{c}{ct_9+d}.$$

# $g = 0$ GW potential of $\mathbb{P}_{4,4,2}$

$$
\begin{aligned}
\mathcal{F}_0^{\mathbb{P}_{4,4,2}} &= \frac{1}{2}t_1^2 t_9 + t_1\left(\frac{t_2 t_4}{4} + \frac{t_3^2}{8} + \frac{t_5 t_7}{4} + \frac{t_6^2}{8} + \frac{t_8^2}{4}\right) \\
&+ \frac{x^\infty}{8}\left(t_3 t_2^2 + t_5^2 t_6\right) + \frac{y^\infty}{8}\left(t_6 t_2^2 + t_3 t_5^2\right) + \frac{z^\infty}{4} t_2 t_5 t_8 \\
&- w^\infty\Bigg(\frac{t_3^4}{128} + \frac{1}{64}t_6^2 t_3^2 + \frac{1}{32}t_8^2 t_3^2 + \frac{1}{32}t_2 t_4 t_3^2 + \frac{1}{32}t_5 t_7 t_3^2 + \frac{t_6^4}{128} \\
&+ \frac{t_8^4}{32} + \frac{1}{64}t_2^2 t_4^2 + \frac{1}{64}t_2^2 t_4^2 + \frac{1}{32}t_2 t_4 t_6^2 + \frac{1}{32}t_5^2 t_7^2 + \frac{1}{32}t_6^2 t_8^2 + \frac{1}{16}t_2 t_4 t_8^2 \\
&+ \frac{1}{16}t_5 t_7 t_8^2 + \frac{1}{32}t_5 t_6^2 t_7 + \frac{1}{16}t_2 t_4 t_5 t_7\Bigg) \\
&+ \frac{x^\infty z^\infty}{16}\left(t_3 t_4 t_5 t_8 + t_2 t_6 t_7 t_8\right) + \frac{y^\infty z^\infty}{16}\left(t_4 t_5 t_6 t_8 + t_2 t_3 t_7 t_8\right) + \dots
\end{aligned}
$$

# $\mathrm{SL}(2,\mathbb{C})$–acted $g=0$ GW potential of $\mathbb{P}_{4,4,2}$

$$\left(\mathcal{F}_0^{\mathbb{P}_{4,4,2}}\right)^A = \frac{1}{2}t_1^2t_9 + t_1\left(\frac{t_2t_4}{4} + \frac{t_3^2}{8} + \frac{t_5t_7}{4} + \frac{t_6^2}{8} + \frac{t_8^2}{4}\right)$$

$$+ \frac{x^A}{8}\left(t_3t_2^2 + t_5^2t_6\right) + \frac{y^A}{8}\left(t_6t_2^2 + t_3t_5^2\right) + \frac{z^A}{4}t_2t_5t_8$$

$$- w^A\left(\frac{t_3^4}{128} + \frac{1}{64}t_6^2t_3^2 + \frac{1}{32}t_8^2t_3^2 + \frac{1}{32}t_2t_4t_3^2 + \frac{1}{32}t_5t_7t_3^2 + \frac{t_6^4}{128}\right.$$

$$+ \frac{t_8^4}{32} + \frac{1}{64}t_2^2t_4^2 + \frac{1}{64}t_2^2t_4^2 + \frac{1}{32}t_2t_4t_6^2 + \frac{1}{32}t_5^2t_7^2 + \frac{1}{32}t_6^2t_8^2 + \frac{1}{16}t_2t_4t_8^2$$

$$\left.+ \frac{1}{16}t_5t_7t_8^2 + \frac{1}{32}t_5t_6^2t_7 + \frac{1}{16}t_2t_4t_5t_7\right)$$

$$+ \frac{x^Az^A}{16}\left(t_3t_4t_5t_8 + t_2t_6t_7t_8\right) + \frac{y^Az^A}{16}\left(t_4t_5t_6t_8 + t_2t_3t_7t_8\right) + \dots$$

# Primitive form change, simple–elliptic singularities

$$\mathcal{A}^{(\tau_0,\omega_0)} := \begin{pmatrix} \dfrac{\bar{\tau}_0}{4\pi\omega_0 \mathrm{Im}(\tau_0)} & \omega_0\tau_0 \\ \dfrac{1}{4\pi\omega_0 \mathrm{Im}(\tau_0)} & \omega_0 \end{pmatrix} \quad \text{for some } \tau_0 \in \mathbb{H}, \omega_0 \in \mathbb{C}^*$$

Proposition[B.–Takahashi].

The action of $\mathcal{A}^{(\tau_0,\omega_0)}$ above, combined with rescaling, is equivalent to the primitive form change.

For two primitive forms $\zeta_1$ and $\zeta_2$, $\exists \tau_0, \omega_0$ s.t.

$$\mathcal{F}^{W,\zeta_1}(\mathbf{t}) = \left(\mathcal{F}^{W,\zeta_2}\right)^{\mathcal{A}^{(\tau_0,\omega_0)}}(\widetilde{\mathbf{t}}).$$

where $\widetilde{\mathbf{t}} = \widetilde{\mathbf{t}}(\mathbf{t})$ is the rescaling of the variables.

# Orbit and special points

Assume

$$\mathcal{F}_{\widetilde{E}_7}^{(\tau_0,\omega_0)} := \left(\mathcal{F}_0^{\mathbb{P}_{4,4,2}}\right)^{\mathcal{A}^{(\tau_0,\omega_0)}}.$$

Mirror symmetry for a fixed $\tilde{E}_7$ = classify $\mathcal{F}_{\widetilde{E}_7}^{(\tau_0,\omega_0)}$ for all possible $\tau_0$ and $\omega_0$.

# Orbit and special points

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Theorem(B.-Priddis)

Take $\tau_1 := \sqrt{-1}$ and $\omega_1 := \sqrt{2\pi}/(\Gamma(\frac{3}{4}))^2$. Then $\mathcal{F}_{\widetilde{E}_7}^{(\tau_1,\omega_1)}$ coincides with the genus zero potential of FJRW theory of $\widetilde{E}_7$ with maximal symmetry group up to a linear change of the variables.

$$
\begin{aligned}
\mathcal{F}^{(\sqrt{-1},\omega_1)}_{\tilde{E}_7}(\tilde{\mathbf{t}}) = {} & \frac{1}{2}\tilde{t}_{11}^2\tilde{t}_{33} + \tilde{t}_{11}\left(\frac{\tilde{t}_{22}^2}{2} + \tilde{t}_{21}\tilde{t}_{23} + \tilde{t}_{13}\tilde{t}_{31} + \tilde{t}_{12}\tilde{t}_{32}\right) \\
& - \tilde{t}_{12}^2\tilde{t}_{13}\left(\frac{\tilde{t}_{33}}{8} + \frac{\tilde{t}_{33}^5}{61440}\right) - \tilde{t}_{21}^2\tilde{t}_{31}\left(\frac{\tilde{t}_{33}}{8} + \frac{\tilde{t}_{33}^5}{61440}\right) \\
& + \tilde{t}_{13}\tilde{t}_{21}^2\left(\frac{1}{2} + \frac{\tilde{t}_{33}^4}{3072} + \frac{\tilde{t}_{33}^8}{330301440}\right) \\
& + \tilde{t}_{12}^2\tilde{t}_{31}\left(\frac{1}{2} + \frac{\tilde{t}_{33}^4}{3072} + \frac{\tilde{t}_{33}^8}{330301440}\right) + \dots
\end{aligned}
$$

$$\mathcal{F}_{\tilde{E}_7}^{(\sqrt{-1},\omega_1)}(\tilde{\mathbf{t}}) = \frac{1}{2}\tilde{t}_{11}^2\tilde{t}_{33} + \tilde{t}_{11}\left(\frac{\tilde{t}_{22}^2}{2} + \tilde{t}_{21}\tilde{t}_{23} + \tilde{t}_{13}\tilde{t}_{31} + \tilde{t}_{12}\tilde{t}_{32}\right)$$
$$- \tilde{t}_{12}^2\tilde{t}_{13}\left(\frac{\tilde{t}_{33}}{8} + \frac{\tilde{t}_{33}^5}{61440}\right) - \tilde{t}_{21}^2\tilde{t}_{31}\left(\frac{\tilde{t}_{33}}{8} + \frac{\tilde{t}_{33}^5}{61440}\right)$$
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## Conjecture(B.)

There is non–trivial $A_0 \in \mathrm{SL}(2,\mathbb{Z})$, s.t.

$$\mathcal{F}_{\widetilde{E}_7}^{(\tau_1,\omega_1)} = \left(\mathcal{F}_{\widetilde{E}_7}^{(\tau_1,\omega_1)}\right)^{A_0}.$$

This is the only pair $(\tau_1,\omega_1)$ among all $(\tau_0,\omega_0)$, s.t. such an element $A_0$ exists and $\mathcal{F}_{\widetilde{E}_7}^{(\tau_1,\omega_1)} \in \mathbb{Q}[[\mathbf{t}]]$.