# Hyperelliptic sigma functions in mathematical physics.

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# Elliptic sigma functions.

The standard Weierstrass model for a plane elliptic curve is

$$V = \{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - g_2 x - g_3\}. \tag{1}$$

The discriminant of this curve is $\mathcal{D} = g_2^3 - 27g_3^2$.

The curve $V$ is non-degenerate curve when $\mathcal{D} \neq 0$.

Set

$$2\omega_k = \oint_{a_k} \frac{dx}{y}, \qquad 2\eta_k = -\oint_{a_k} \frac{xdx}{y}, \qquad k = 1, 2, \tag{2}$$

where $\frac{dx}{y}$ and $\frac{xdx}{y}$ are basic differentials and $a_k$ are the basic cycles on the curve such that

$$\eta_1\omega_2 - \omega_1\eta_2 = \frac{\pi i}{2}.$$

A non-degenerate curve $V$ defines a lattice $\Gamma \subset \mathbb{Z}^2 \subset \mathbb{C}$ of rank $2$ generated by $2\omega_1$, $2\omega_2$, with Im $\frac{\omega_2}{\omega_1} > 0$.

An elliptic function is a meromorphic function on $\mathbb{C}$ such that

$$f(z + 2\omega_k) = f(z), \quad k = 1, 2.$$

That is, it can be considered as a function on a complex torus $T = \mathbb{C}/\Gamma$.

The torus $T$ is known as the Jacobian $Jac(V)$ of the curve $V$.

# Weierstrass $\wp$-function.

Weierstrass $\wp$-function is the unique elliptic function $\wp(z) = \wp(z; g_2, g_3)$ on $\mathbb{C}$ with poles only in lattice points such that $\lim_{z\to 0} \left(\wp(z) - \frac{1}{z^2}\right) = 0$.

The function $\wp(z)$ is an even function and all its poles are double poles. It defines a uniformization of the standard elliptic curve:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

## Corollary.

The function $u(z) = 2\wp(z)$ is a solution of the stationary KdV equation

$$u''' = 6uu'.$$

# Weierstrass $\zeta$-function.

Weierstrass $\zeta$-function is the odd meromorphic function $\zeta(z) = \zeta(z; g_2, g_3)$ such that

$$\zeta'(z) = -\wp(z) \qquad \text{and} \qquad \lim_{z\to 0}\left(\zeta(z) - \frac{1}{z}\right) = 0.$$

The periodic properties of $\zeta'(z)$:

$$\zeta(z + 2\omega_k) = \zeta(z) + 2\eta_k,$$

and we have $\eta_k = \zeta(\omega_k)$, $k = 1, 2$.

# Weierstrass $\sigma$-function.

Weierstrass $\sigma$-function is the entire odd function $\sigma(z) = \sigma(z; g_2, g_3)$ such that

$$\big(\ln \sigma(z)\big)' = \zeta(z) \qquad \text{and} \qquad \lim_{z\to 0}\left(\frac{\sigma(z)}{z}\right) = 1.$$

The periodic properties of $\sigma(z)$:

$$\sigma(z + 2\omega_k) = -\sigma(z)\exp\big(2\eta_k(z + \omega_k)\big), \quad k = 1, 2.$$

The initial segment of the series has the form

$$\sigma(z) = z - \frac{g_2}{2}\frac{z^5}{5!} - 6g_3\frac{z^7}{7!} - 9\frac{g_2^2}{4}\frac{z^9}{9!} - 18g_2g_3\frac{z^{11}}{11!} + \dots$$

# Lamé equation.

The elliptic Baker–Akhiezer function

$$\Phi(z) = \Phi(z;\alpha) = \frac{\sigma(\alpha - z)}{\sigma(z)\sigma(\alpha)} \exp\big(\zeta(\alpha)z\big) \tag{3}$$

represents a solution of Lamé equation

$$\left(\frac{d^2}{dz^2} - 2\wp(z)\right)\Phi(z) = \wp(\alpha)\Phi(z). \tag{4}$$

# Vector fields tangent to the discriminant.

Consider the fields on $\mathbb{C}^2$ with coordinates $g_2$ and $g_3$

$$l_0 = 4g_2 \frac{\partial}{\partial g_2} + 6g_3 \frac{\partial}{\partial g_3}, \quad l_2 = 6g_3 \frac{\partial}{\partial g_2} + \frac{1}{3} g_2^2 \frac{\partial}{\partial g_3}.$$

We have $[l_0, l_2] = 2l_2$. Since $l_0\mathcal{D} = 12\mathcal{D}$ and $l_2\mathcal{D} = 0$, the fields $l_0$ and $l_2$ are tangent to the discriminant manifold $\{(g_2, g_3) \in \mathbb{C}^2 \ : \ \mathcal{D}(g_2, g_3) = 0\}$.

## Integral curves of the field $l_0$.

Let $\tau$ be the natural parameter on the curves defined by the dynamical system $\left(l_0 = \frac{\partial}{\partial \tau}\right)$

$$g_2' = 4g_2, \quad g_3' = 6g_3.$$

Then $g_2(\tau) = g_2(0)\exp(4\tau), \quad g_3(\tau) = g_3(0)\exp(6\tau)$.

# Integral curves of the field $l_2$.

Let $t$ be the natural parameter on the curves defined by the dynamical system $\left(l_2 = \frac{\partial}{\partial t}\right)$

$$g_2' = 6g_3, \quad g_3' = \frac{1}{3}g_2^2.$$

Then

$$g_2(t) = 3\wp(t+d; 0, b_3), \qquad g_3(t) = \frac{1}{2}\wp'(t+d; 0, b_3),$$

where $b_3 = \frac{4}{27}\mathcal{D}(g_2(0), g_3(0))$ and $d$ is the solution of the compatible system of equations

$$\wp(d; 0, b_3) = \frac{1}{3}g_2(0), \quad \wp'(d; 0, b_3) = 2g_3(0).$$

# The Weierstrass theorem.

Let

$$l_0 = 4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3}, \qquad l_2 = 6g_3\frac{\partial}{\partial g_2} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3},$$

$$H_0 = z\frac{\partial}{\partial z} - 1, \qquad H_2 = \frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{24}g_2 z^2,$$

$$Q_0 = H_0 - l_0, \qquad Q_2 = H_2 - l_2.$$

## Theorem.

The operators $Q_0$ and $Q_2$ annihilate the sigma-function:

$$Q_0\sigma(z; g_2, g_3) = 0, \qquad Q_2\sigma(z; g_2, g_3) = 0. \tag{5}$$

The system of equations (5) in the nonholonomic frame $\left(\frac{\partial}{\partial z}, l_0, l_2\right)$ uniquely determines the $\sigma$-function $\sigma(z; g_2, g_3)$ as the solution with the initial condition $\sigma(z; 0, 0) = z$.

# The heat equation.

## Corollary.

The function $\psi(z,t)$ such that

$$\psi(z,t) = \exp\big(h(t)z^2 + r(t)\big)\sigma\big(z, g_2(t), g_3(t)\big) \tag{6}$$

for some functions $r(t)$, $h(t)$, $g_2(t)$ and $g_3(t)$ satisfies the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t) \tag{7}$$

if and only if the functions $r(t)$, $h(t)$, $g_2(t)$ and $g_3(t)$ satisfy the homogeneous polynomial dynamical system in $\mathbb{C}^4$ with coordinates $h$, $r$, $g_2$, $g_3$ :

$$h' = 2h^2 - \frac{1}{24}g_2, \qquad r' = 3h, \tag{8}$$

$$g_2' = 6g_3 + 8hg_2, \qquad g_3' = \frac{1}{3}g_2^2 + 12hg_3. \tag{9}$$

# Chazy equation.

## Theorem.

The functions $r(t)$, $h(t)$, $g_2(t)$ and $g_3(t)$ satisfy the dynamical system (8)–(9) if and only if $h(t)$ satisfies Chazy equation

$$h''' - 24hh'' + 36(h')^2 = 0, \tag{10}$$

and

$$g_2 = -24(h' - 2h^2), \quad g_3 = -4(h'' - 12h'h + 16h^3), \quad r' = 3h.$$

For initial data

$$h_0 = h(0), \quad h_1 = h'(0), \quad h_2 = h''(0)$$

there exists the unique solution of the Chazy equation (10).

## Corollary.

For given $(h_0, h_1, h_2)$ there exists the unique up to a factor solution of the heat equation (7) of the form (6).

# Burgers equation.

Consider Burgers equation

$$v_t + v v_z = \frac{1}{2} v_{zz}. \tag{11}$$

Cole-Hopf transform of the function $\psi(z,t)$ is

$$v(z,t) = -\frac{\partial \ln \psi(z,t)}{\partial z}.$$

The following identity holds:

$$v_t + v v_z - \tfrac{1}{2} v_{zz} = -\tfrac{\partial}{\partial z} \left( \tfrac{\psi_t - \frac{1}{2}\psi_{zz}}{\psi} \right).$$

**Corollary.**

Let $\psi(z,t)$ be a solution of the heat equation $\psi_t = \frac{1}{2}\psi_{zz}$ or the equation $\psi_t = \frac{1}{2}\psi_{zz} + \lambda\psi$ where $\lambda = const$.

Then $v(z,t)$ is a solution of Burgers equation (11).

# A solution of Burgers equation.

Let $\psi(z,t) = \exp\big(h(t)z^2 + r(t)\big)\sigma\big(z, g_2(t), g_3(t)\big)$, as before, be a solution of the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t).$$

## Theorem.

The function

$$v(z,t) = -\frac{\partial \ln \psi(z,t)}{\partial z} = -2h(t)z - \zeta\big(z; g_2(t), g_3(t)\big)$$

gives a solution of Burgers equation. Here $h(t)$ is a solution of Chazy equation

$$h''' - 24hh'' + 36(h')^2 = 0,$$

and $g_2$, $g_3$ are given by

$$g_2 = 24(2h^2 - h'), \quad g_3 = -4(h'' - 12h'h + 16h^3).$$

# Klein problem.

The problem of construction of multidimensional $\sigma$-functions is a classic one.
In 1886, F.Klein proposed the following problem:
Modify multidimensional $\theta$-function $\theta(\mathbf{u}; \Gamma_V)$ in order to obtain an entire function which is:

(1) independent of a choice of basis in the lattice $\Gamma_V$;
(2) covariant with respect to the Möbius transformations of the curve $V$.

On this problem, Klein published 3 works (1886–1890).

In 1923, a 3-volume collection of Klein's scientific works was published.
In the preface to the works on the problem under discussion, he emphasized that the theory of hyperelliptic functions is still far from complete.

The covariance requirement (2) immediately led to the need to confine ourselves to the class of hyperelliptic curves. But even this case caused artificial difficulties.

In his work (1903), H.F.Baker disregarded requirement (2) and showed that in the case of curves of genus 2, it is possible to construct analogues of elliptic $\sigma$-functions without using $\theta$-functions of genus 2.

Since 1990, in a cycle of works, V.Buchstaber, V.Enolskii and D.Leykin have been developed a theory of multidimensional $\sigma$-functions associated with given models of plane algebraic curves.

# Our setting of problem.

Let $V_\lambda = \left\{(x,y) \in \mathbb{C}^2 \ : f(x,y;\lambda) = 0\right\}$ be a plane nonsingular algebraic curve of genus $g$, where $f(x,y;\lambda)$ is a polynomial in $x$ and $y$ with the coefficient vector $\lambda = (\lambda_{q_1}, \ldots, \lambda_{q_d}) \in \mathcal{B}_g$, where $\mathcal{B}_g$ is an open dense subset in $\mathbb{C}^d$.

Denote by $\Gamma_\lambda \subset \mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ the lattice of periods of holomorphic differentials on $V_\lambda$.

## Problem.

Construct an entire function $\sigma(\mathbf{u};\lambda)$ such that:

(1) The decomposition of $\sigma(\mathbf{u};\lambda)$ at the origin is given by the series in $\mathbf{u}$ with polynomial coefficients in $\lambda$.

(2) For any $k_1$ and $k_2$, the function $\wp_{k_1,k_2}(\mathbf{u};\lambda) = -\frac{\partial}{\partial u_{k_1}}\frac{\partial}{\partial u_{k_2}} \ln \sigma(\mathbf{u};\lambda)$ defines a meromorphic function on $Jac(V_\lambda) = \mathbb{C}^g/\Gamma_\lambda$.

(3) The function $\sigma(\mathbf{u};0)$ is a polynomial of a given form.

In the case of hyperelliptic curves of genus $g$, in our works (1997-1999) we constructed the desired $\sigma$-function $\sigma(\mathbf{u};\lambda)$, where

$$\mathbf{u} = (u_1, \dots, u_{2g-1}), \quad \lambda = (\lambda_4, \dots, \lambda_{4g+2}).$$

On its basis we obtained the key results of the theory of hyperelliptic functions of genus $g > 1$. (see below)

In subsequent works, for coprime $n$ and $s$, $n < s$, we introduced a class of plane $(n,s)$-curves of genus $\frac{(n-1)(s-1)}{2}$ with the coefficient vector $(\lambda_{q_1}, \dots, \lambda_{q_d})$, where $d = \frac{(n+1)(s+1)}{2} - \left[\frac{s}{n}\right] - 3$, and constructed the corresponding $\sigma$-functions.

A family $V_\lambda$ of $(n,s)$-curves is defined by the polynomials

$$f(x,y;\lambda) = y^n - x^s - \sum_{i,j} \lambda_{q(i,j)} x^i y^j,$$

where $0 \leqslant i < s-1$, $0 \leqslant j < n-1$ and $q(i,j) = ns - in - js > 0$.

Hyperelliptic curves of genus $g$ are $(2, 2g+1)$-curves. The trigonal curves $(3,s)$ have genus $g = 3m$ at $s = 3m+1$ and genus $g = 3m+1$ at $s = 3m+2$, $m > 0$.

# Hyperelliptic curves.

Consider the curve

$$V_\lambda = \left\{ (x, y) \in \mathbb{C}^2 \, : y^2 = f(x; \lambda) = x^{2g+1} + \sum_{k=2}^{2g+1} \lambda_{2k} x^{2g-k+1} \right\}, \qquad (12)$$

where $g \geqslant 1$ and $\lambda = (\lambda_4, \ldots, \lambda_{4g+2}) \in \mathbb{C}^{2g}$ are the parameters.

Set $M_\mathcal{D} = \{\lambda \in \mathbb{C}^{2g} \, : f(x; \lambda)$ has multiple roots$\}$ and $\mathcal{B} = \mathcal{B}_g = \mathbb{C}^{2g} \setminus M_\mathcal{D}$.

For any $\lambda \in \mathcal{B}$ we have the Jacobian variety $Jac(V_\lambda) = \mathbb{C}^g / \Gamma_\lambda$ and the field of meromorphic functions $F_\lambda$ on $Jac(V_\lambda)$.

# Arnold's convolution.

Let $g \in \mathbb{N}$. Consider the space $\mathbb{C}^{2g+1}$ with coordinates $\xi = (\xi_1, \ldots, \xi_{2g+1})$. Let

$$H = \{\xi \in \mathbb{C}^{2g+1} \; : \; \sum_{k=1}^{2g+1} \xi_k = 0\}.$$

The permutation group $S_{2g+1}$ of coordinates of $\mathbb{C}^{2g+1}$ corresponds to the action of the group $A_{2g} \subset S_{2g+1}$ on $H$. We associate with each vector $\xi \in H$ the polynomial

$$\prod_k (x + \xi_k) = x^{2g+1} + \lambda_4 x^{2g-1} + \lambda_6 x^{2g-2} + \ldots + \lambda_{4g} x + \lambda_{4g+2}.$$

We obtain $\mathbb{C}^{2g} \cong H/A_{2g}$ with coordinates $(\lambda_4, \ldots, \lambda_{4g+2})$.

Let $\pi \colon H \to \mathbb{C}^{2g}$ be the canonical projection and $a = a(\lambda)$, $b = b(\lambda)$ be functions on $\mathbb{C}^{2g}$.

## Definition. Arnold's convolution

The formula $\pi^*(a * b) = \langle \nabla \pi^* a, \nabla \pi^* b \rangle$ gives the function $a * b = (a * b)(\lambda)$, where $\langle \cdot, \cdot \rangle$ is Euclidean scalar product, and $\nabla$ is the gradient in $\mathbb{C}^{2g}$.

# Polynomial Lie algebra.

Set $T_{2k,2m}(\lambda) = \lambda_{2k} * \lambda_{2m}$, $2 \leqslant k, m \leqslant 2g+1$, and

$$\ell_{2i} = \sum_{s=2}^{2g+1} T_{2i+2,2s-2} \frac{\partial}{\partial \lambda_{2s}}, \quad i = 0, \ldots, 2g-1.$$

We have obtained $2g$-polynomial vector fields on $\mathbb{C}^{2g}$ that are linearly independent at any point of $\mathcal{B} = \mathbb{C}^{2g} \setminus M_D$ and are tangent to the discriminant hypersurface $M_D$.

The vector fields $\{\ell_{2i},\ i = 0, \ldots, 2g-1\}$ generate the graded polynomial Lie algebra $\mathcal{L}_g$.

Let $c_{2i,2j}^{2s}(\lambda)$ be the structural polynomials of $\mathcal{L}_g$. We have

$$[\ell_{2i}, \ell_{2j}] = \sum_{s=0}^{2g-1} c_{2i,2j}^{2s}(\lambda) \ell_{2s}.$$

# The hyperelliptic $\sigma$-function of genus $g$.

## Theorem.

For any $g \geqslant 1$, there exists the function $\sigma(\mathbf{u}, \lambda)$ such that:

(a) $\sigma(\mathbf{u}, \lambda)$ is an entire quasiperiodic function of $\mathbf{u} \in \mathbb{C}^g$ and $\lambda \in \mathcal{B} = \mathbb{C}^{2g} \setminus M_D$.

(b) $\partial_{2i-1}\partial_{2j-1} \log \sigma(\mathbf{u}, \lambda) = -\wp_{2i-1,2j-1}(\mathbf{u}, \lambda) \in F_\lambda$ whenever $\lambda \in \mathcal{B}$, where $\partial_{2i-1} = \frac{\partial}{\partial u_{2i-1}}$ and $i, j = 1, \ldots, g$.

(c) $\sigma(\mathbf{u}; 0)$ coincides with Adler-Moser polynomial up to a constant factor.

(d) $\sigma(\mathbf{u}, \lambda)$ is a solution of the system $Q_{2j}\sigma(\mathbf{u}, \lambda) = 0$, $j = 0, \ldots, 2g-1$, where $Q_{2j} = \ell_{2j} - \frac{1}{2}H_{2j} - \delta_{2j}(\lambda)$, with $\ell_{2j} \in \mathcal{L}_g$ and

$$H_{2j} = \alpha_j^{kl}(\lambda)\partial_{2k-1}\partial_{2l-1} + 2\beta_{jk}^l(\lambda)u_{2k-1}\partial_{2l-1} + \gamma_{jkl}(\lambda)u_{2k-1}u_{2l-1},$$

$$\delta_{2j}(\lambda) = \frac{1}{8}\ell_{2j} \log\det T(\lambda) + \frac{1}{2}\beta_{jk}^k(\lambda),$$

where the summation from 1 to $2g$ extends over the repeated indices. Here $\alpha_j^{kl}(\lambda) = \alpha_j^{lk}(\lambda)$, $\beta_{jk}^l(\lambda)$ and $\gamma_{jkl}(\lambda) = \gamma_{jlk}(\lambda)$ are polynomials of $\lambda$.

# The annihilators $Q_{2j}$ of the $\sigma$-function and a quantum oscillator.

Write the system of equations $Q_{2j}\sigma(\mathbf{u},\lambda) = 0,\ j = 1,\dots,2g$, in the form of Schrödinger equations $\ell_{2j}(\sigma) = \left\{\frac{1}{2}H_{2j} + \delta_{2j}(\lambda)\right\}\sigma$,

of a multidimensional quantum harmonic oscillator with multiple 'times'.

The formalism of quantum oscillator:

$H_{2j}$ is a set of 'quadratic Hamiltonians',

$\ell_{2j}$ are derivatives over 'times',

$\delta_{2j}$ is 'the energy of an oscillator mode'.

The realization of the sigma-function in the form of an average of the 'ground state wave-function' (a multidimensional Gaussian function) over a lattice suggests a natural interpretation of sigma-function as the 'wave-function of the coherent state' of the oscillator.

# Lie algebras of Schrödinger operators.

Let $Q_{2j} = \ell_{2j} - \frac{1}{2}H_{2j} - \delta_{2j}(\lambda)$ be our Schrödinger operators.

## Theorem.

The operators $Q_{2j}$, $j = 0, \dots, 2g-1$, on functions in $\mathbf{u}$ and $\lambda$ satisfy the commutation relations

$$[Q_{2i}, Q_{2j}] = \sum_{s=0}^{2g-1} c_{2i,2j}^{2s}(\lambda) Q_{2s}$$

where $c_{2i,2j}^{2s}(\lambda)$ are the structural polynomials of the polynomial Lie algebra $\mathcal{L}_g$.

We denote by $\pi\colon \mathcal{U}_g \to \mathcal{B}_g$ the universal bundle of Jacobian varieties $J_\lambda = Jac(V_\lambda)$ of hyperelliptic curves.
Let us consider the mapping $\varphi\colon \mathcal{B}_g \times \mathbb{C}^g \to \mathcal{U}_g$ such that

$$\begin{array}{ccc} \mathcal{B}_g \times \mathbb{C}^g & \xrightarrow{\varphi} & \mathcal{U}_g \\ \downarrow & & \downarrow \\ \mathcal{B}_g & = & \mathcal{B}_g \end{array},$$

which defines the projection $\lambda \times \mathbb{C}^g \to \mathbb{C}^g/\Gamma_\lambda$ for any $\lambda \in \mathcal{B}_g$.

We denote by $\mathcal{F} = \mathcal{F}_g$ the field of functions on $\mathcal{U}_g$ such that for any $f \in \mathcal{F}$ the function $\varphi^*(f)$ is meromorphic, and its restriction to the fiber $J_\lambda$ is an Abelian function for any point $\lambda \in \mathcal{B}_g$, e.i. $f(\mathbf{u} + 2\Omega) = f(\mathbf{u})$ for any $2\Omega \in \Gamma_\lambda$.

Let $\mathbf{u} = \mathbf{t}$, $t_1 = x$, $f'(\mathbf{t}) = \frac{\partial}{\partial x} f(\mathbf{t})$, $\partial_{2k-1} = \frac{\partial}{\partial t_{2k-1}}$, and

$$\omega = \begin{pmatrix} j_1 & \dots & j_s \\ 2k_1 - 1 & \dots & 2k_s - 1 \end{pmatrix}$$

where $1 \leqslant k_1 < \ldots < k_s$, $1 \leqslant s \leqslant g$, $j_q > 0$, $q = 1, \ldots, s$, and $j_1 + \ldots + j_s \geqslant 2$. Set

$$\wp_\omega = \wp_\omega(\mathbf{t}) = -\partial_{2k_1-1}^{j_1} \cdots \partial_{2k_s-1}^{j_s} \ln \sigma(\mathbf{t}). \tag{13}$$

Let $\mathcal{P}$ denote the subring over $\mathbb{Q}$ in the field $\mathcal{F}$ generated by the functions $\wp_\omega$ for all $\omega$ described above.

## Theorem.

There exists an isomorphism

$$\mathcal{P} = \mathbb{Q}\left[\wp(1), \ldots, \wp(g)\right],$$

where $\wp(k) = (\wp_{2k}, \wp'_{2k}, \wp''_{2k})$ and $\wp_{2k} = -\partial_1 \partial_{2k-1} \ln \sigma(\mathbf{t})$.

Let $\wp_{2i-1,2k-1} = \wp_{2i-1,2k-1}(\mathbf{t}) = -\partial_{2i-1}\partial_{2k-1} \ln \sigma(\mathbf{t})$ where $i \neq 1$ or $k \neq 1$.

**Theorem.**

All algebraic relations between hyperelliptic functions $\wp_\omega$ of genus $g$ follow from the relations which, in our notations, have the form

$$\wp_{2i}'' = 6(\wp_{2i+2} + \wp_2\wp_{2i}) - 2(\wp_{3,2i-1} - \lambda_{2i+2}\delta_{i,1}). \tag{14}$$

$$\begin{aligned}\wp_{2i}'\wp_{2k}' = 4(\wp_{2i}\wp_{2k+2} + \wp_{2i+2}\wp_{2k} + \wp_2\wp_{2i}\wp_{2k} + \wp_{2i+1,2k+1}) - \\ - 2(\wp_{2i}\wp_{3,2k-1} + \wp_{2k}\wp_{3,2i-1} + \wp_{2i-1,2k+3} + \wp_{2i+3,2k-1}) + \\ + 2(\lambda_{2i+2}\wp_{2k}\delta_{i,1} + \lambda_{2k+2}\wp_{2i}\delta_{k,1}) + 2\lambda_{2(i+j+1)}(2\delta_{i,k} + \delta_{i,k-1} + \delta_{i-1,k}).\end{aligned} \tag{15}$$

Here $\delta_{i,k}$ is the Kronecker symbol, $\deg \delta_{i,k} = 0$.

## Corollary.

For all $g \geqslant 1$, we have the following relations:

1. Setting $i = 1$ in (14), we obtain

$$\wp_2'' = 6\wp_2^2 + 4\wp_4 + 2\lambda_4. \tag{16}$$

2. Setting $i = 2$ in (14), we obtain

$$\wp_4'' = 6(\wp_2\wp_4 + \wp_6) - 2\wp_{3,3}. \tag{17}$$

3. Setting $i = k = 1$ in (15), we obtain

$$(\wp_2')^2 = 4\left[\wp_2^3 + (\wp_4 + \lambda_4)\wp_2 + \wp_{3,3} - \wp_6 + \lambda_6\right]. \tag{18}$$

We have $\wp_{2i}' = \partial_{2i-1}\wp_2$. Then from (16) we obtain:

## Corollary.

For any $g > 1$, the function $u = 2\wp(\mathbf{t})$ is a solution of KdV equation

$$u''' = 6uu' + 4\dot{u}, \quad \text{where} \quad \dot{u} = 2\partial_3\wp_2.$$

## Theorem.

The projection of the universal bundle

$$\pi_g : \mathcal{U}_g \to \mathcal{B}_g \subset \mathbb{C}^{2g}$$

is given by the polynomials $\lambda_{2k} = \lambda_{2k}\big(\wp(1), \dots, \wp(g)\big) \in \mathcal{P},\ k = 2, \dots, 2g+1$ of degree at most 3.

Examples.

1. From (16), we obtain

$$\lambda_4 = \frac{1}{2}\wp_2'' - 3\wp_2^2 - 2\wp_4.$$

2. From (17) and (18), we obtain

$$\lambda_6 = \frac{1}{4}(\wp_2')^2 - \left[\wp_2^3 + (\wp_4 + \lambda_4)\wp_2 + \wp_{3,3} - \wp_6\right]$$

where $\wp_{3,3} = 3(\wp_2\wp_4 + \wp_6) - \frac{1}{2}\wp_4''$.

The field $\mathcal{F} = \mathcal{F}_g$ contains the coordinate ring $\Lambda = \mathbb{Q}[\lambda_4, \dots, \lambda_{4g+2}]$ of the parameter space $\mathcal{B}_g$.

Denote by $\mathcal{P}_\Lambda$ the algebra of polynomials over $\Lambda$ generated by the hyperelliptic function $\wp_2$ and all its derivatives with respect to $x$.

## Theorem.

For all $k \geqslant 1$, the hyperelliptic functions $\wp_{2k}$ belong to the ring $\mathcal{P}_\Lambda$, i.e. there exist differential polynomials $\Psi_{2k}(\wp_2, \wp_2', \dots)$ over the ring $\Lambda$ such that

$$2\wp_{2k} = \Psi_{2k}(\wp_2, \wp_2', \dots).$$

Examples.

1. From (16), we obtain

$$4\wp_4 = \wp_2'' - 6\wp_2^2 - 2\lambda_4.$$

Thus $$2\wp_4 = \Psi_4 = \frac{1}{2}(\wp_2'' - 6\wp_2^2 - 2\lambda_4).$$

2. From (17) and (18), we obtain the polynomial $\Psi_6$ using the system

$$6\wp_6 - 2\wp_{3,3} = \wp_4'' - 6\wp_2\wp_4;$$
$$4\wp_6 - 4\wp_{3,3} = 4\wp_2^3 + 4(\wp_4 + \lambda_4)\wp_2 - (\wp_2')^2 + 4\lambda_6.$$

# Korteweg-de Vries equation.

$$\text{KdV}: \ U_t = 6UU_x - U_{xxx}, \quad \text{where } U = U(x,t).$$

Set:
$$x = t_1, \qquad \partial_1 = \frac{\partial}{\partial t_1},$$
$$t = t_3, \qquad \partial_3 = \frac{\partial}{\partial t_3} \qquad \Phi_2 = U(t_1, t_3).$$

There is a differential conservation law

$$\partial_3 \Phi_2 \ = \ \partial_1 \Phi_4, \quad \text{where } \ \Phi_4 = 3\Phi_2^2 - \partial_1^2 \Phi_2.$$

The equation $\partial_3 \Phi_2 \ = \ 0$ is the stationary KdV equation where $\Phi_2 = U(t_1)$ and we obtain $\partial_1^2 \Phi_2 = 3\Phi_2^2 + \alpha_4$.

In the case $\Phi_2 = U(t_1 - \alpha_2 t_3)$ we have the equation of travelling wave

$$\partial_1^2 \Phi_2 = 3\Phi_2^2 + \alpha_2 \Phi_2 + \alpha_4.$$

Here $\alpha_2$ and $\alpha_4$ are constants.

# Lenard operator and differential polynomials.

Let $U = U(\mathbf{t})$ be an infinitely differentiable function, $\mathbf{t} = (t_1, t_3, \ldots, t_{2k-1}, \ldots)$. Set $\partial = \frac{\partial}{\partial t_1}$ and $f'(\mathbf{t}) = \partial f(\mathbf{t})$.

**Definition.**

The operator $L = -\partial^2 + 2u + u'\partial^{-1}$ is called Lenard operator.

Set $\Phi_2 = U$ and define $\Phi_{2k}$, $k > 1$, by the recursion

$$\partial \Phi_{2k+2} = L \partial \Phi_{2k}. \tag{19}$$

Example. $\Phi_4 = 3U^2 - U''$.

**Theorem.**

Formula (19) defines the differential polynomials $\Phi_{2k}(U, U', \ldots, U^{(2k-2)})$.

Example. $\Phi_6 = 10U^3 - 10UU'' - 5(U')^2 + U^{(4)}$.

# Hierarchy of Korteweg-de Vries equation.

Let $U(\mathbf{t})$ be a solution of KdV equation

$$U''' = 6UU' - \dot{U}$$

where $U'(\mathbf{t}) = \partial_1 U(\mathbf{t})$ and $\dot{U}(\mathbf{t}) = \partial_3 U(\mathbf{t})$.

**Theorem.**

Set $U(\mathbf{t}) = \Phi_2$. Then the family of differential conservation laws

$$\partial_{2k-1}\Phi_2 = \partial_1\Phi_{2k}, \; k = 2, 3, \ldots$$

is equivalent to the hierarchy of KdV equation for the function $U(\mathbf{t})$.

# Hierarchy of Korteweg-de Vries equation in the class of functions on $g$ variables.

Let us consider a meromorphic function $u(t_1, \ldots, t_{2g-1})$ on $\mathbb{C}^g$.
Dependence on higher times is given by the formula

$$U(\mathbf{t}) = u\Big(t_1 - \sum_{k>g} \alpha_{1,2k-1} t_{2k-1}, \ldots, t_{2g-1} - \sum_{k>g} \alpha_{2g-1,2k-1} t_{2k-1}\Big),$$

where $\alpha_{i,j}$ are constants.

## Theorem.

Hierarchy

$$\partial_{2k-1}\Phi_2 = \partial_1\Phi_{2k}, \; k = 2, 3, \ldots$$

is equivalent to $g$-stationary hierarchy of KdV equation for the function $U(\mathbf{t})$ and $\Phi_{2k}$ are as above.

We have $2\wp_{2k}(\mathbf{t};\lambda) = \Psi_{2k}(\wp_2, \wp_2', \ldots)$, where $\mathbf{t} = (t_1, \ldots, t_{2g-1})$ and $k = 1, \ldots, g$ (see slide 28).

## Theorem.

For any $g \geqslant 1$, the hyperelliptic function of genus $g$

$$u(t_1, \ldots, t_{2g-1}) = 2\wp_2(\mathbf{t};\lambda)$$

satisfies $g$-stationary hierarchy of KdV equation.

## Corollary.

There exists the formula

$$\Psi_{2k} = \sum_{i=0}^{k} \alpha_{2i}\Phi_{2k-2i}$$

where $\alpha_0 = (-1)^{k-1}4^{k-1}$ and $\alpha_{2i} \in \Lambda = Q[\lambda_4, \ldots, \lambda_{4g+2}]$.

Example. $\Psi_4 = -4\Phi_4 - \lambda_4$.

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THANK YOU FOR ATTENTION!