1. Artin algebras and deformation functors

Let k be a field. Recall that a commutative algebra A over k is called *Artinian* if any descending chain of ideals stabilizes. An Artin algenra A is called local if it is a local ring with residue field k, we denote by $\mathfrak{m}_A \subset A$ the maximal ideal of A.

Remark 1.1. Algebra A is Artinian iff $\dim A < \infty$ and A is local.

Remark 1.2. Every Artin algebra is Noetherian.

Lemma 1.3. *We have $\mathfrak{m}_A^N = 0$ for $N \gg 0$.*

Proof. Indeed by Krull's intersection theorem we have $\bigcap_{i \in \mathbb{Z}_{\geqslant 0}} \mathfrak{m}_A^i = 0$ and now the claim follows from the descending property applied to the ideals $\mathfrak{m}_A^i$. $\square$

We denote by $\mathbf{Art}_k$ the category of local Artin k-algebras A with residue field k. The main example for us will be $A = k[\epsilon]/\epsilon^2$.

Definition 1.4. *A deformation functor is a covariant functor $D\colon \mathbf{Art}_k \to \mathbf{Set}$ such that $D(k) = \{*\}$, where $\{*\}$ is some fixed one-element set.*

Let us give some examples of deformation functors.

Definition 1.5. *For any scheme X we define a functor of points $h_X\colon \mathbf{Art}_k \to \mathbf{Set}$ given by $A \mapsto \mathrm{Maps}(\mathrm{Spec}\, A, X)$. If $X = \mathrm{Spec}\, R$ then $h_R(A) = \mathrm{Mor}(R, A)$.*

Definition 1.6. *Let $B \twoheadrightarrow A$ be a surjection with kernel K, $A, B \in \mathbf{Art}_k$. We say*

$$0 \to K \to B \to A \to 0$$

is a small extension if $\mathfrak{m}_B K = 0$.

The simplest example is $A = k$, $B = k[\varepsilon]/\varepsilon^2$, $K = (\varepsilon)$.

Remark 1.7. Let N be a module over A, we can associate to it the following ring: $A * N$, as a vector space we have $A * N := A \oplus N$ and the ring structure is the following: $(a_1, n_1)(a_2, n_2) = (a_1 a_2, a_1 n_2 + a_2 n_1)$. Note that $N \subset A * N$ is an ideal such that $N^2 = 0$. Note also that $A * N = A \oplus \varepsilon N/(\varepsilon^2)$. Extensions $A * N$ are called trivial.

Lemma 1.8. *Any surjection $B \twoheadrightarrow A$ can be obtained as a composition of small extensions.*

Proof. Recall that $K \subset \mathfrak{m}_B$ is the kernel of our surjection $\pi\colon B \twoheadrightarrow A$. Note that by lemma 1.3 there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $\mathfrak{m}_B^N = 0$, hence, $K^N = 0$. Set $B_i := B/K^i$, $i = 1, \dots, N$ and note that $B_0 = A$, $B_N = B$. It follows from the definitions that the natural surjections $B_i \twoheadrightarrow B_{i-1}$ define small extensions. Now $B \twoheadrightarrow A$ is the composition $B = B_N \twoheadrightarrow B_{N-1} \twoheadrightarrow \dots \twoheadrightarrow B_1 \twoheadrightarrow B_0 = A$. $\square$

Let R be a complete local k-algebra such that $\dim_k(\mathfrak{m}_R/\mathfrak{m}_R^2) = d < \infty$.

Lemma 1.9. *There exists a surjection $k[[x_1, \dots, x_d]] \twoheadrightarrow R$.*

Proof. Recall that R is complete so we have the isomorphism $R \xrightarrow{\sim} \varprojlim R/\mathfrak{m}_R^N$. It is enough to construct compatible surjections $k[[x_1, \dots, x_d]] \twoheadrightarrow R/\mathfrak{m}_R^N$. This is an easy inductive construction. □

Consider now a small extension $B \twoheadrightarrow A$. Let us analyze the morphism $h_R(B) = \mathrm{Mor}(R, B) \to \mathrm{Mor}(R, A) = h_R(A)$.

1. Assume that $R = k[[x_1, \dots, x_d]]$. The morphism $h_R(B) \to h_R(A)$ is surjective since if $f \in \mathrm{Mor}(R, A)$ is some morphism and $[\tilde{x}_i] := f(x_i)$ then let $\tilde{x}_i \in B$ be any liftings of $[\tilde{x}_i]$ and we can define $\tilde{f} \colon R \to B$ by $\tilde{f}(x_i) = \tilde{x}_i$. It remain to describe the fibers of the morphism $h_R(B) \twoheadrightarrow h_R(A)$. Note that two functions $\tilde{f}_1, \tilde{f}_2 \in \mathrm{Mor}(R, B)$ restrict to the same function $f \in \mathrm{Mor}(R, A)$ iff the image of $\tilde{f}_1 - \tilde{f}_2$ lies in $K \subset B$. It is easy to deduce from $K^2 = 0$ that $\tilde{f}_1 - \tilde{f}_2 \colon R \to K$ is a derivation over k. Here the structure of R-module on K comes from the homomorphism $f \colon R \to A$ and the natural action $A \curvearrowright K$ (again use that $K^2 = 0$). To show this we just write

$$
\begin{aligned}
(\tilde{f}_1 - \tilde{f}_2)(ab) &= \tilde{f}_1(a)\tilde{f}_1(b) - \tilde{f}_2(a)\tilde{f}_2(b) = \tilde{f}_1(a)\tilde{f}_2(b) + \tilde{f}_1(b)\tilde{f}_2(a) - 2\tilde{f}_2(a)\tilde{f}_2(b) = \\
&= \tilde{f}_2(b)(\tilde{f}_1(a) - \tilde{f}_2(a)) + \tilde{f}_2(a)(\tilde{f}_1(b) - \tilde{f}_2(b)) = b \cdot (\tilde{f}_1 - \tilde{f}_2)(a) + a \cdot (\tilde{f}_1 - \tilde{f}_2)(b),
\end{aligned}
$$

in the second equality we use that $(\tilde{f}_1 - \tilde{f}_2)(a)(\tilde{f}_1 - \tilde{f}_2)(b) = 0$ since $K^2 = 0$. Note now that $\tilde{f}_1 - \tilde{f}_2$ is uniquely determined by its restriction to $\mathfrak{m}_S$ and moreover $(\tilde{f}_1 - \tilde{f}_2)(\mathfrak{m}_R^2) \subset \mathfrak{m}_B K = 0$ so we conclude that $\tilde{f}_1 - \tilde{f}_2$ is uniquely determined by its restriction to $\mathfrak{m}_R/\mathfrak{m}_R^2$. We obtain the following exact sequence

$$\mathrm{Hom}(\mathfrak{m}_R/\mathfrak{m}_R^2, K) \to h_R(B) \to h_R(A) \to 0$$

Note that we have the identification $\mathrm{Hom}(\mathfrak{m}_R/\mathfrak{m}_R^2, K) = (\mathfrak{m}_R/\mathfrak{m}_R^2)^* \otimes K$.

2. Consider now the case of general R. By Lemma (1.9) we have a surjection $\pi \colon S := k[[x_1, \dots, x_d]] \twoheadrightarrow R$ which induces an isomorphism $\mathfrak{m}_S/\mathfrak{m}_S^2 \xrightarrow{\sim} \mathfrak{m}_R/\mathfrak{m}_R^2$. We set $I := \ker \pi$ and note that $I \subset \mathfrak{m}_S^2$. In the same way as above it is easy to see that the kernel of the map $\mathrm{Mor}(R, B) \to \mathrm{Mor}(R, A)$ identifies with $\mathrm{Hom}(\mathfrak{m}_R/\mathfrak{m}_R^2, K) = (\mathfrak{m}_R/\mathfrak{m}_R)^* \otimes K$. We can now define some space and a morphism ob such that $f \in h_R(A)$ has a lifting $\tilde{f}$ iff $\mathrm{ob}(f) = 0$. Pick $f \in h_R(A)$ and consider the morphism $g := f \circ \pi$. Note that the morphism $\mathrm{Mor}(S, B) \to \mathrm{Mor}(S, A)$ is surjective. The lifting $\tilde{f}$ exists iff there exists a lifting $\tilde{g} \in \mathrm{Mor}(S, B)$ such that $\tilde{g}|_I = 0$. Note now that for any two liftings $\tilde{g}_1, \tilde{g}_2 \in \mathrm{Mor}(S, B)$ the morphism $\tilde{g}_1 - \tilde{g}_2$ is a derivative and $I \subset \mathfrak{m}_S^2$ so $\tilde{g}_1|_I = \tilde{g}_2|_I$. We conclude that the morphism $f \mapsto \tilde{g}|_I \in \mathrm{Hom}(I/\mathfrak{m}_S I, K)$. So we obtain an exact sequence

$$\mathrm{Hom}(\mathfrak{m}_R/\mathfrak{m}_R^2, K) \to h_R(B) \to h_R(A) \to \mathrm{Hom}(I/\mathfrak{m}_S I, K).$$

Note that $\mathfrak{m}_R/\mathfrak{m}_R^2$ is the fiber at the point $\mathfrak{m}_R$ of the cotangent sheaf Ω_R and $(I/\mathfrak{m}_S I)^*$ is exactly the fiber of normal sheaf $\mathcal{N}_{\mathcal{M}/Y}$, where $\mathcal{M} := \operatorname{Spec} R$, $Y := \operatorname{Spec} S$.

Remark 1.10. In general if we have a closed embedding $\mathcal{M} \hookrightarrow Y$ of some scheme into the affine scheme then obstructions for $\mathcal{M}$ are $\mathcal{N}_{\mathcal{M}/Y}$ and deformations are controled by $\mathcal{T}_{\mathcal{M}}$.

Note that there exists a natural morphism $\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{N}_{\mathcal{M}/Y}$ which kernel is exactly $\mathcal{N}_{\mathcal{M}/Y}$. We have a morphism of complexes $[\mathcal{T}_{\mathcal{M}} \xrightarrow{0} \mathcal{N}_{\mathcal{M}/Y}] \to [\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{N}_{\mathcal{M}/Y}]$ in degrees -1, 0 which an isomorphism at the level of -1 and surjective at 0 cohomology. This is a general property for obstruction theories to have a morphism to a so called tangent complex which can be represented in derived category by the complex $[\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{N}_{\mathcal{M}/Y}]$ (note that cohomologies of $[\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{N}_{\mathcal{M}/Y}]$ does not depend on the embedding of $\mathcal{M}$ into Y).

1.1. Let us now discuss local structure in terms of tangent/obstruction complex.

Let us now discuss the relation between a local structure of a moduli functor $\mathcal{M}\colon \mathbf{Sch} \to \mathbf{Set}$ at some point $w \in \mathcal{M}(k)$ and its tangent-obstruction $T_w\mathcal{M} \xrightarrow{0} \operatorname{Ob}_w \mathcal{M}$. There exists the natural homomorphism of algebras $\hat{S}^\bullet((\operatorname{Ob}_w \mathcal{M})^*) \to \hat{S}^\bullet((T_w\mathcal{M})^*)$ called Kuranishi map. We will construct it later.

Remark 1.11. Recall that for a vector space E over k we denote by $\hat{S}^\bullet(E)$ the inverse limit $\bigoplus_{k=0}^{k=n} S^k(N)$.

Let us denote by $\mathcal{M}^\wedge$ the formal neighbourhood of the point w i.e. a functor $\mathrm{Art}_k \to \mathrm{Set}$ given by $A \mapsto \mathcal{M}(A) \times_{\mathcal{M}(A/\mathfrak{m}_A)} \{w\}$.

Remark 1.12. Here we use the isomorphism $k \xrightarrow{\sim} A/\mathfrak{m}$ to obtain an identification $\mathcal{M}(k) \xrightarrow{\sim} \mathcal{M}(A/\mathfrak{m}_A)$ so w defines a point $w \in \mathcal{M}(A/\mathfrak{m}_A)$ and $\mathcal{M}^\wedge(A)$ is a fiber over $\{w\}$ of the natural morphism $\mathcal{M}(A) \to \mathcal{M}(A/\mathfrak{m}_A)$. Note that when $\mathcal{M}$ is represented by a scheme $\operatorname{Spec} R$ and $\mathfrak{m} \subset R$ is the maximal ideal corresponding to a point w then $\mathcal{M}^\wedge$ is represented by $R^\wedge := \lim R/\mathfrak{m}^N$. This follows from the fact that the homomorphism from $R^\wedge$ to Artin A is the same as a homomorphism from R to A such that $\mathfrak{m}$ maps to $\mathfrak{m}_A$.

We assume that the functor $\mathcal{M}$ is (locally) represented by a scheme M and there is a tangent/obstruction theory on $\mathcal{M}$ which induces on $\mathcal{M}^\wedge$ our tangent/obstruction for $\operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{M},w}$ (as above).

Proposition 1.13. *The functor $\mathcal{M}^\wedge$ is represented by the scheme $\operatorname{Spec}(\hat{S}^\bullet(T_w^*)) \otimes_{\hat{S}^\bullet(\operatorname{Ob}_w^*)} k$.*

Proof. We can assume that $\mathcal{M}$ is represented by some scheme X of finite type. Note that the question is local so we can assume that $X = \operatorname{Spec} R$, where R is a complete local ring such that $\dim_k(\mathfrak{m}_R/\mathfrak{m}_R^2) = d < \infty$. Recall that in this case we have $T = \operatorname{Hom}(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$, $\operatorname{Ob} = \operatorname{Hom}(I/I\mathfrak{m}_S, K)$. Let us for simplicity assume that $K = k$ so we have $T^* = \mathfrak{m}_R/\mathfrak{m}_R^2$, $\operatorname{Ob}^* = I/I\mathfrak{m}_S$. Then the Kuranishi map

$$\hat{S}^\bullet(I/I\mathfrak{m}_S) \to \hat{S}^\bullet(\mathfrak{m}_R/\mathfrak{m}_R^2) = S$$

is induced by the natural embedding $I \hookrightarrow S$. Note now that $S \otimes_{\hat{S}^\bullet(I/I\mathfrak{m}_S)} k$ is exactly $S/I \simeq R$ and the claim follows. Let us be more accurate here. Let $f_1, \ldots, f_n$

be generators of $I \subset S$. Let $[f_i] \in I/I\mathfrak{m}_S$ be a basis in $I/I\mathfrak{m}_S$ and $f_i \in I$ be any representatives. We claim that f_i generate I as a module over S. This is an immedeate consequence of the Nakayama lemma. Now our morphism K is induced by the morphism $I/I\mathfrak{m}_S \to S$, $[f_i] \mapsto f_i$.

□

Let us now make an important comment. Recall that a functor F is called formally smooth if the morphism $F(B) \to F(A)$ is surjective for any simple extension $0 \to K \to B \to A \to 0$. It is a standard fact that if F is represented by a scheme X of a finite type then the formal smoothness of F is equivalent to the smoothness of X. Note now that formal smoothness implies that there are no obstructions so by Proposition 1.13 locally F should be isomorphic to $T_w F$. This is exactly the case for smooth schemes. So we actually already know Proposition 1.13 for functors F with no obstructions (formally smooth functors).

2. Obstructions for G/H

Recall that the fiber of Ob at the point $(C, p_1, \dots, p_n, f) \in \mathcal{M}_{0,n,\beta}(X)$ is $H^1(C, f^*TX)$. Variety X is called *convex* if Ob $= 0$. Goal of this section is to show that homogeneous spaces are convex (in particular, flag varieties) so the corresponding moduli spaces of stable maps are smooth.

Proposition 2.1. *Let G be an algebraic group and $H \subset G$ is a subgroup. Then for $X = G/H$ the vector bundle TX is globally generated.*

Proof. The action $G \curvearrowright X$ induces a morphism $\inf\colon \mathfrak{g} \to \Gamma(X, k)$. It follows from the fact that the action of G is transitive that the image of inf generates TX. □

Proposition 2.2. *Assume that X is such that TX is globally generated. Then X is convex.*

Proof. Consider some morphism $\mu\colon C \to X$. Our goal is to show that $H^1(f^*TX, C) = 0$. Note that f^*TX is globally generated so we have a short exact sequence

$$0 \to K \to \Gamma(f^*TX, C) \otimes \mathcal{O}_C \to f^*TX \to 0$$

which induces the long exact sequence

$$\to \Gamma(f^*TX, C) \otimes H^1(C, \mathcal{O}_C) \to H^1(C, f^*TX) \to H^2(C, K) \to .$$

Note now that $H^2(C, K) = 0$ since $\dim C = 1$ and $H^1(C, \mathcal{O}_C) = H^0(C, \omega_C) = H^0(C, \mathcal{O}_C(-2)) = 0$. We conclude that $H^1(C, f^*TX) = 0$. □

Remark 2.3. Note that every irreducible component of C is smooth (it is isomorphic to $\mathbb{P}^1$) so the dualizing complex of C is $\omega_C[1] = \mathcal{O}_C(-2)[1]$.

3. Main example of section of a vector bundle

Starting from a variety $\mathcal{M}$ together with a closed embedding $\iota\colon \mathcal{M} \hookrightarrow \mathbb{A}^m$ given by some functions $(f_1, \dots, f_n)$ we can associate to it the following complex of sheaves on $\mathcal{M}$:

$$\mathcal{T}_{\mathbb{A}^m}|_{\mathcal{M}} \xrightarrow{\sigma} \mathcal{O}_{\mathcal{M}}^{\oplus n},$$

where σ is induced by $d\iota$. Let us now give the algebraic description of the morphism σ. We give the description of the dual morphism $\sigma^*\colon \mathcal{O}_{\mathcal{M}}^{\oplus n} \to \mathcal{T}_{\mathbb{A}^m}^{\vee}|_{\mathcal{M}}$. Let S be the ring of global functions on $\mathbb{A}^m$ and R be the ring of global functions on X. We have the natural surjection $S \twoheadrightarrow R$ with kernel I generated by $f_1, \dots, f_n$. Note that $\mathcal{T}_{\mathbb{A}^n}^*|_{\mathcal{M}} = R \otimes_S \Omega_S$. It follows from the definitions that the motphism $\sigma^*\colon R^{\oplus n} \to R \otimes_S \Omega_S$ is induced by the composition $R^{\oplus n} \to I \to R \otimes_S \Omega_S$ where the first morphism sends $(0, \dots, 1, \dots, 0)$ to f_i and the second morphism is given by $f \mapsto 1 \otimes df$.

Remark 3.1. In slightly more general situation when we have a closed embedding of $\mathcal{M} \hookrightarrow Y$ as zeroes of some section of some vector bundle $E \to Y$ then we always have a morphism $TY|_{\mathcal{M}} \to E|_{\mathcal{M}}$ induced by the composition $TY|_{\mathcal{M}} \to N_{\mathcal{M}/Y} \hookrightarrow E|_{\mathcal{M}}$.

Fix now a point $x \in \mathcal{M}$. Our goal is to describe obstructions at this point. So we can assume that R, S are local and complete with maximal ideals corresponding to x. We claim that the obstructions lie in $O_x := \operatorname{coker} \sigma_x$. Consider the morphism $F\colon k^n \to S = \hat{S}^{\bullet}(\mathfrak{m}_S/\mathfrak{m}_S^2)$ given by $(0, , \dots, 1, \dots, 0) \mapsto f_i$. Note now that the induced morphism $k^n \to \mathfrak{m}_S/\mathfrak{m}_S^2$ coincides with σ_x^* after the identification $(R \otimes_S \Omega_S)_x = k \otimes_R (R \otimes_S \Omega_S) = k \otimes_S \Omega_S = \mathfrak{m}_S/\mathfrak{m}_S^2$. Note also that the image of the morphism F generates an ideal I.

Let us now start from a simple extension of Artin rings

$$0 \to K \to B \to A \to 0$$

and consider a morphism $\varphi\colon S/(F) \to A$. We want extend it to a morphism $S/(F) \to B$. Note that we always have an extension $g\colon S \to B$ of the composition $S \twoheadrightarrow S/(F) \to A$. It follows from the definitions that the morphism $k^{\oplus n} \xrightarrow{F} S \xrightarrow{g} B$ factors through $o\colon k^{\oplus n} \to K$. Let $\operatorname{ob}(\varphi, A, B)$ be the image of o under the obvious morphism $k^{\oplus n} \otimes K \to O \otimes K$. Indeed the condition that $\operatorname{ob}(\varphi, A, B)$ is equivalent to the fact that the fact that $o\colon k^{\oplus n} \to K$ lifts to the morphism $h\colon \mathfrak{m}_S/\mathfrak{m}_S^2 \to K$ and we denote by $\hat{h}\colon S \to B$ the corresponding homomorphism of algebras. It remains to note that $g - \hat{h}$ factors through $R = S/(F) \to B$ extending the morphism φ.

Remark 3.2. The fact that $(g - \hat{h})|_{(F)} = 0$ is equivalent to $g \circ F = \hat{h} \circ F$ but $g \circ F = o$ which lifts to h and the claim follows. The fact that $g - \hat{h}$ extends φ follows from the definition of g and the fact that h maps to K.

So we see that for this exampe the tangent-obstruction complex is homologies of the complex $\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{E}|_{\mathcal{M}}$. Such obstruction theories are called perfect (when they from a complex of locally trivial sheaves) and in these cases one can define virtual fundamental class of $\mathcal{M}$ (in dimension $\operatorname{rk} \mathcal{E}^0 - \operatorname{rk} \mathcal{E}^1$ where $\mathcal{E}^0 \to \mathcal{E}^1$ is our obstruction theory).

Note also that we have a natural morphism $[\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{N}_{\mathcal{M}/Y}] \to [\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{E}|_{\mathcal{M}}]$ induced by the morphisms $\mathcal{T}_Y|_{\mathcal{M}} \xrightarrow{\mathrm{Id}} \mathcal{T}_Y|_{\mathcal{M}}$, $\mathcal{N}_{\mathcal{M}/Y} \xrightarrow{s} \mathcal{E}|_{\mathcal{M}}$ and this morphism is an isomorphism on 0 cohomologies and is injective on 1st cohomologies (people standatdly consider dual complexes).

4. Virtual fundamental class in our main example

Let $\mathcal{M}$ be the moduli functor of virtual dimension n (virtual dimension depends on the (perfect) tangent/obstruction theory on $\mathcal{M}$). Our goal is to define a fundamental class of $\mathcal{M}$ which will be an element of $A_n(\mathcal{M})$ (the n-th Chow group). There is a homomorphism $A_n(\mathcal{M}) \to H_{2n}^{BM}(\mathcal{M}, \mathbb{C})$ so one can think about our fundamental class as about an element of Borel-Moore homology.

Remark 4.1. Let us recall the definition of Chow groups of scheme $\mathcal{M}$. This is a quotient of the group (over $\mathbb{C}$) generated by subvarieties of $\mathcal{M}$ by rational equivalences i.e. if we have a subvariety $W \subset \mathcal{M}$ of dimension $i+1$ and a rational function f on W then the divisor (f) of f should be equal to zero in $A_i(\mathcal{M})$.

Remark 4.2. We recall that if $\mathcal{M}$ is a topological space then the Borel-Moore homology $H_\bullet^{BM}(\mathcal{M}, \mathbb{C})$ can be defined as $H_\bullet(\overline{\mathcal{M}}, \overline{\mathcal{M}} \setminus \mathcal{M})$, where $\overline{\mathcal{M}}$ is any compactification of $\mathcal{M}$ (for example, one-point compactification).

Remark 4.3. Let us finally remark that for (possibly noncompact) manifold $\mathcal{M}$ there exists an isomorphism $H_i^{BM}(\mathcal{M}, \mathbb{C}) \simeq H^{\dim \mathcal{M} - i}(M, \mathbb{C})$.

4.1. Notations/definitions. Let us recall come notations. Let $\mathcal{M} \hookrightarrow Y$ be a closed embedding of schemes and $\mathcal{I}$ is the ideal definig $\mathcal{M}$. Then we define $\mathcal{N}_{\mathcal{M}/Y} := (\mathcal{I}/\mathcal{I}^2)^\vee$, this is a sheaf on $\mathcal{M}$. We denote by $N_{\mathcal{M}/Y} \to \mathcal{M}$ the corresponding scheme over $\mathcal{M}$ which is defined as the relative spectrum $\operatorname{Spec} S^\bullet_{\mathcal{O}_\mathcal{M}}(\mathcal{I}/\mathcal{I}^2)$.

Remark 4.4. There is a general way to pass from a coherent sheaf $\mathcal{E}$ on a scheme X to the correspondong total space $E \to X$ over X. Assume first that $X = \operatorname{Spec} A$ is an affine scheme. Then we have $\mathcal{E} = \tilde{M}$ for some A-module M ($M = \Gamma(X, \mathcal{E})$). Now we can define $E := \operatorname{Spec}(S^\bullet_A(M))$, here $S^\bullet_A(M) := \bigoplus_{i \geqslant 0} S^i_A(M)$ and $S^i_A(M)$ is the quotient of the tensor product $M \otimes_A M \otimes_A \ldots \otimes_A M$ and $S^0_A(M) = A$. Note that we have the natural morphism $E \to X$, which corresponds to the embedding $A = S^0(M) \hookrightarrow S^\bullet_A(M)$. We can apply the same construction to the sheaf $\mathcal{E}$ (using some affine covering of X) and get the desired scheme E. Note that if $\mathcal{E}$ is locally trivial then E is the total space of the vector bundle corresponding to $\mathcal{E}^\vee$. Note also that the construction is functorial in the following sence. If $\mathcal{E}$, $\mathcal{P}$ are coherent sheaves on X and $E \to X$, $P \to X$ are the corresponding total spaces. Then any morphism $\mathcal{E} \to \mathcal{P}$ over $\mathcal{O}_X$ induces a morphism $P \to E$ of schemes over X (clear from the construction).

In more details: morphism $\mathcal{E} \to \mathcal{P}$ induces a morphism the morphism $S^\bullet_{\mathcal{O}_X}(E) \to S^\bullet_{\mathcal{O}_X}(P)$ and so a morphism $P = \operatorname{Spec}_{\mathcal{O}_X} S^\bullet_{\mathcal{O}_X}(\mathcal{P}) \to \operatorname{Spec}_{\mathcal{O}_X} S^\bullet_{\mathcal{O}_X}(\mathcal{E}) = E$.

We have the natural morphism of sheaves $\mathcal{I}/\mathcal{I}^2 \to \Omega_Y|_\mathcal{M}$ which induces the morphism $TY|_\mathcal{M} \to N_{\mathcal{M}/Y}$. Note that when the embedding $\mathcal{M} \hookrightarrow Y$ is regular then the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally trivial and we have the identification $\mathcal{I}/\mathcal{I}^2 \xrightarrow{\sim} (\mathcal{I}/\mathcal{I}^2)^{\vee\vee}$ so $N_{\mathcal{M}/Y}$ is exactly the normal bundle to $\mathcal{M} \subset Y$.

Remark 4.5. Note that we have a short exact sequence of sheaves on $\mathcal{M}$

$$0 \to \mathcal{T}_\mathcal{M} \to \mathcal{T}_Y|_\mathcal{M} \to \mathcal{N}_{\mathcal{M}/Y}$$

and the morphism $\mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{N}_{\mathcal{M}/Y}$ is surjective when $\mathcal{M}$ is smooth. In general we can define $\mathcal{T}^1_{\mathcal{M}}$ to be the cokernel of this morphism. This sheaf is called a *tangent sheaf* and it actually does not depend on the embedding $\mathcal{M} \hookrightarrow Y$. So in general we have the following exact sequence of sheaves

$$0 \to \mathcal{T}_{\mathcal{M}} \to \mathcal{T}_Y|_{\mathcal{M}} \to \mathcal{N}_{\mathcal{M}/Y} \to \mathcal{T}^1_{\mathcal{M}} \to 0.$$

4.2. Simple simple case. We start from the simplest case. Let E be a vector bundle of rank r on a smooth variety A of dimension n and $s \in \Gamma(A, E)$. Assume that $\mathcal{M} = Z(s)$ the zero locus of s.

Recall that if $P \to M$ is a (real) vector bundle of rank k over a (smooth) variety M then we have a canonically defined Euler class $\mathrm{eu}(P) \in H^k(M, \mathbb{C})$ such that the Poincare dual class $\mathrm{PD}_M(\mathrm{eu}(P)) \in H_{\dim M - k}(M, \mathbb{C})$ is the class of zeroes of a generic section of P.

Let us return to our example. Recall that $\operatorname{vdim} M = \dim Y - \operatorname{rk} E$. The simplest case is when s is generic so M is smooth and $\dim \mathcal{M} = \operatorname{vdim} \mathcal{M}$. Then we already know that we must have $[\mathcal{M}]^{vir} := [\mathcal{M}]$ and the last can be thought as $\mathrm{PD}_{\mathcal{M}}(\mathrm{eu}(0))$. Note that in terms of $\sigma\colon TY|_{\mathcal{M}} \to E|_{\mathcal{M}}$ this happens exactly when σ is surjective (i.e. $\mathrm{Ob} = 0$).

Assume now that $\mathcal{M}$ is still smooth but has the wrong dimension (for example $s = 0$ and $M = Y$). This is the case when section s takes values in some vector subbundle $E' \subset E$ of rank r' and is transverse to the zero section of E' and (in C^∞-setting) we have a decomposition $E = E' \oplus (E/E')$. We see that $s = (s', 0)$ for some $s' \in \Gamma(A, E')$. Note that we can now deform $(s', 0)$ to a section (s', ϵ), $\epsilon \in \Gamma(A, E/E')$ with ϵ transverse to the zero section of E/E' and we see that our virtual class should be a class of generic section on the bundle E/E' i.e. can be defined as follows $[\mathcal{M}]^{vir} := \mathrm{PD}_{\mathcal{M}}(\mathrm{eu}((E/E')|_{\mathcal{M}}))$.

Note now that the sheaf $(E/E')|_{\mathcal{M}}$ canonically identifies with $\operatorname{coker} \sigma = \mathrm{Ob}$. So we conclude that in this case we define $[\mathcal{M}]^{vir} := \mathrm{PD}_{\mathcal{M}}(\mathrm{eu}(\mathrm{Ob}))$. This suggests us the following definition when Ob is a vector bundle over $\mathcal{M}$.

$$[\mathcal{M}]^{vir} := \mathrm{PD}_{\mathcal{M}}(\mathrm{eu}(\mathrm{Ob})).$$

Let us check that this class (in our example) will lie in the correct dimension (i.e. in dimension $\operatorname{vdim} \mathcal{M} = \dim Y - \operatorname{rk} E$). Indeed we have an exact sequence

$$0 \to T\mathcal{M} \to TY|_{\mathcal{M}} \to E \to \mathrm{Ob} \to 0$$

so $\operatorname{rk} \mathrm{Ob} = \operatorname{rk} E + \dim M - \dim Y$ i.e. $\mathrm{eu}(\mathrm{Ob}) \in H^{2 \operatorname{rk} E + 2 \dim M - 2 \dim Y}(\mathcal{M}, \mathbb{C})$. It follows that $\mathrm{PD}(\mathrm{eu}(\mathrm{Ob})) \in H_{2 \dim Y - 2 \operatorname{rk} E}(\mathcal{M}, \mathbb{C})$ and that is it.

Recall that we want to intersect zero section of $E \to Y$ with a transverse section of E. Topologically we know that the answer is always $\mathrm{PD}_Y(\mathrm{eu}\, E)$ but the problem is that this is a class in cohomologies of Y not of $\mathcal{M}$. So in general we want to construct some element $[\mathcal{M}]^{vir} \in H_{2 \operatorname{vdim} \mathcal{M}}(\mathcal{M}, \mathbb{C})$ such that its push-forward to $H_{2 \operatorname{vdim} \mathcal{M}}(\mathcal{M}, \mathbb{C})$ is $\mathrm{PD}_Y(\mathrm{eu}\, E)$. So we want to deform $\mathcal{M}$ inside itself. The idea is not to deform $\mathcal{M}$ inside Y but to deform Y itself (to a certain cone) such that this deformation is trivial being restricted to $\mathcal{M} \hookrightarrow Y$. There is a canonical construction of such a deformation called a *deformation to normal cone*.

4.3. Rees construction and normal cone. Let us now discuss some important general construction (called Rees construction). Let A be a commutative algebra over $\mathbb{C}$. Assume that we are given an increasing filtration $\ldots \subset A_{-1} \subset A_0 \subset A_1 \subset$ on the algebra A such that $A_i \cdot A_j \subset A_{i+j}$. We can form the following algebra $A_\hbar$ over $\mathbb{C}[\hbar]$, $A_\hbar := \bigoplus_i \hbar^i A_i$ and note that this is indeed an algebra with respect to the multiplication $a_i\hbar^i \cdot a_j\hbar^j = a_ia_j\hbar^{i+j}$ and $\hbar^i \in A_\hbar$ for any $i \geqslant 0$ since $1 \in A_0 \subset A_i$. We obtain the family $\pi\colon \operatorname{Spec} A_{\bar{\hbar}} \to \operatorname{Spec}(\mathbb{C}[\bar{\hbar}])$. Let us first of all note that $A_\hbar$ is $\hbar$-graded so the morphism is actually $\mathbb{C}^\times$-equivariant with respect to the standard action $\mathbb{C}^\times \curvearrowright \mathbb{A}^1$. We conclude that for any $t \neq 0$ the fiber $\pi^{-1}(t)$ identifies with $\pi^{-1}(1)$. Let us compute $\pi^{-1}(1)$. By the definition it is the spectrum of the quotient $\bigoplus_i \hbar^i A_i/(\hbar - 1)$ i.e. equals to $\sum_i A_i = A$. Let us now compute the fiber $\pi^{-1}(0)$. By the definition it is the spectrum of the quotient $\bigoplus_i \hbar^i A_i/(\hbar)$ and it is easy to see that this algebra naturally identifies with $\bigoplus_i A_i/A_{i+1}$. We conclude that the family π deforms $\operatorname{Spec} A$ to $\operatorname{Spec} \operatorname{gr} A$.

Starting from a closed embedding $\mathcal{M} \hookrightarrow Y$ with defining ideal sheaf $\mathcal{I}$ we can define a filtration in $\mathcal{O}_Y$ by $\mathcal{I}^i$, where $\mathcal{I}^i = \mathcal{O}_Y$ for $i \leqslant 0$. In other words we should consider

$$\ldots \oplus \hbar^{-2}\mathcal{I}^2 \oplus \hbar^{-1}\mathcal{I} \oplus \mathcal{O}_Y \oplus \hbar\mathcal{O}_Y \oplus \ldots.$$

So we obtain the family $\tilde{Y} \to \mathbb{A}^1$ such that the fiber over $t \neq 0$ is isomorphic to Y and the fiber over 0 is $C_{\mathcal{M}/Y}$ $(= \operatorname{Spec}(\oplus_{i\geqslant 0}\mathcal{I}^i/\mathcal{I}^{i+i}))$. Note also that we have the embedding $\mathcal{M}\times\mathbb{A}^1 \hookrightarrow \tilde{Y}$ corresponding to the ideal $\bigoplus_{i<0} \hbar^i\mathcal{I}^i$ (the quotient by this ideal is natually $\mathcal{O}_{\mathcal{M}}[\hbar]$).

So we see that Y naturally deforms to $C_{\mathcal{M}/Y}$ together with the embedding $\mathcal{M} \hookrightarrow Y$. It is natural to assume that intersection does not depend on the deformation so we can pass from the embedding $\mathcal{M} \hookrightarrow Y$ to the embedding $\mathcal{M} \hookrightarrow C_{\mathcal{M}/Y}$.

4.4. General case of simple case. Recall that we denote by $\mathcal{I} \subset \mathcal{O}_N$ the ideal sheaf of $\mathcal{M}$ and $C_{\mathcal{M}/Y} := \operatorname{Spec}(\bigoplus_{i=0}^\infty I^i/I^{i+1})$ and note that we have the natural surjection $\bigoplus_{i=0}^\infty S^i(\mathcal{I}/\mathcal{I}^2) \twoheadrightarrow \bigoplus_{i=0}^\infty \mathcal{I}^i/\mathcal{I}^{i+1}$ which induces a closed embedding embedding $C_{\mathcal{M}/Y} \hookrightarrow N_{\mathcal{M}/Y}$. We also have the natural embedding $N_{\mathcal{M}/Y} \hookrightarrow E|_{\mathcal{M}}$ (induced by s). Let us now recall how the morphism $N_{\mathcal{M}/Y} \hookrightarrow E|_{\mathcal{M}}$ is constructed.

Recall that $\mathcal{N}_{\mathcal{M}/Y} = (\mathcal{I}/\mathcal{I}^2)^\vee$ and s defines a morphism of sheaves $\mathcal{O}_Y \to \mathcal{E}$ and the dual morphism $s\colon \mathcal{E}^\vee \to \mathcal{O}_Y$. We claim that the image of this morphism is the ideal sheaf $\mathcal{I}$. It is enough to check this locally: when $Y = \operatorname{Spec} S$ and $\mathcal{M} = \operatorname{Spec} R$ are both affine, $\mathcal{E} = \mathcal{O}_Y^{\oplus r}$ and s is then given by r functions $f_i \in S$ and the corresponding morphism $S \to S^{\oplus r}$ sends s to $(sf_1, \ldots, sf_2)$ so the dual morphism $S^{\oplus r} \to S$ sends $(s_1, \ldots, s_r)$ to $f_1s_1 + \ldots + f_rs_r$ and its image is exactly $I = (f_1, \ldots, f_r)$ the ideal of $\mathcal{M} \hookrightarrow Y$.

So we obtain a morphism $\mathcal{E}^\vee \to \mathcal{I}$ which induces a morphism $\mathcal{E}^\vee|_{\mathcal{M}} \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ of sheaves on $\mathcal{M}$. Note that this morphism is surjective! This can be checked locally using the fact that f_i generate the ideal I over S so $[f_i] \in I/I^2$ generate I/I^2 over $R = S/I$.

The surjection $\mathcal{E}^\vee|_{\mathcal{M}} \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ induces the desired embedding $N_{\mathcal{M}/Y} \hookrightarrow E|_{\mathcal{M}}$. One should note that the morphism $TY|_{\mathcal{M}} \to E$ that we have discussed at the previous lecture can be described as follows. It is induced by the morphism of sheaves $\mathcal{E}^\vee \to \Omega_Y|_{\mathcal{M}}$ that is obtained as a composition of the morphisms $\mathcal{E}^\vee \twoheadrightarrow \mathcal{I}/\mathcal{I}^2 \to \Omega_Y|_{\mathcal{M}}$ so (our fundamental map which controls deformations/obstructios) $TY|_{\mathcal{M}} \to E|_{\mathcal{M}}$ is exactly the composition $TY|_{\mathcal{M}} \to N_{\mathcal{M}/Y} \hookrightarrow E|_{\mathcal{M}}$.

4.4.1. *Completing the construction.* So we are in the following situation: We have an embedding $C_{\mathcal{M}/Y} \hookrightarrow E|_{\mathcal{M}}$ and $E|_{\mathcal{M}} \to M$ is the vector bundle over M. We can finally define $[\mathcal{M}]^{vir} := 0^*(C_{\mathcal{M}/Y})$ and we explain in the next section what does it mean.

Note now that in the case when s actually corresponds to some section $(s', 0)$

4.5. Intersection with zero section in vector bundle. Intersection theory for normal cones was developed in Fulton's book and works as follows.

Let us recall the following proposition.

Proposition 4.6. *Let E be a rank r vector bundle on X, $p\colon E \to X$ then the pull-back homomorphism $p^*\colon A_k(X) \to A_{k+r}(E)$ is an isomorphism, here A corresponds to Chow group.*

Remark 4.7. Let us make a comment how to prove the surjectivity of the morphism p^*. We use the induction on the dimension of X. We can always assume that X is irreducible so there exists an open subset $U \hookrightarrow X$ such that $E|_U$ is trivial (so $p^*|_U$ is an isomorphism). Set $Z := X \setminus U$ and denote by $\iota\colon Z \hookrightarrow X$ the closed embedding and by $j\colon U \hookrightarrow X$ the open embedding. We have the following exact sequence

$$A_k(Z) \xrightarrow{\iota_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

and the same exact sequence for $E|_Z$, E, $E|_U$. Now from the surjectivity for $E|_Z$ and $E|_U$ the surjectivity of p^* follows.

Let $s\colon X \hookrightarrow E$ be a zero section and $\alpha \in A_k(E)$ then we can define $s^*(\alpha)$ as the unique cycle in A_{k-r} such that $p^*s^*(\alpha) = \alpha$. So we can now define intersections of closed subvarieties $Z \subset E$ and zero section (of E)! This is exactly what we need.

5. Virtual fundamental class for perfect tangent-obstruction theory

Let us now briefly describe the general aprocach to constructing of $[\mathcal{M}]^{vir}$ starting from a perfect tangent-obstruction theory T.

Let us now define *perfect obstruction theory* on a sheme (more generally Deligne-Mumford stack) X.

It consists of a complex $\mathcal{E}^{-1} \to \mathcal{E}^0$ of locally trivial sheaves on X such that for any embedding $X \hookrightarrow Y$ into a smooth Y we have a morphism of complexes $[\mathcal{E}^{-1} \to \mathcal{E}^0] \to [\mathcal{N}^\vee_{\mathcal{M}/Y} \to \Omega_Y|_{\mathcal{M}}]$ which is an isomorphism on h^0 and surjection on h^{-1}.

Remark 5.1. Note that we have the following exact sequence

$$0 \to (\mathcal{T}^1_{\mathcal{M}})^\vee \to \mathcal{N}^\vee_{\mathcal{M}/Y} \to \Omega_Y|_{\mathcal{M}} \to \Omega_{\mathcal{M}} \to 0$$

so the cohomologies of the complex $[\mathcal{N}^\vee_{\mathcal{M}/Y} \to \Omega_Y|_{\mathcal{M}}]$ do not depend on Y. So we can actually consider $[\mathcal{N}^\vee_{\mathcal{M}/Y} \to \Omega_Y|_{\mathcal{M}}]$ as an object of the derived category $D^b(\mathcal{M})$ (this is a so-called truncated cotangent complex) and define perfect obstruction theory without Y.

In general we construct a vector bundle V on $\mathcal{M}$ and a cone $C \subset V$ and define $[\mathcal{M}]^{vir} := s^*(C)$, where $s\colon \mathcal{M} \hookrightarrow V$ is the zero section. Pair $(C \subset V)$ is constructed purely in terms of tangent-obstruction complex of $\mathcal{M}$. Let us remark that the virtual fundamental class *depends* on the choice of the tangent-obstruction complex.

Let us roughly describe the construction of C and V. Again choose an embedding $\mathcal{M} \hookrightarrow Y$, where Y is smooth (there are no vector bundle E now). We still have the canonical morphisms $\mathcal{I}^k/\mathcal{I}^{k+1} \to S^k(\Omega_Y|_{\mathcal{M}})$ which induces a morphism

$$T_Y|_{\mathcal{M}} \to C_{\mathcal{M}/Y}$$

and we define $C_{\mathcal{M}} := C_{\mathcal{M}/Y}/T_Y|_M$ (this is a stack quotient!). In the same way we can define the quotient $N_{\mathcal{M}}/T_Y|_M$ and note that we have the natural morphism

$$C_{\mathcal{M}} \to N_{\mathcal{M}}.$$

Recall also that we have our perfect obstruction theory $\mathcal{E}^{-1} \to \mathcal{E}^0$ which has a morphism to $\mathcal{N}^{\vee}_{\mathcal{M}/Y} \to \Omega_Y|_{\mathcal{M}}$. They induce morphisms of the corresponding schemes $E_1, E_0, N_{\mathcal{M}/Y}, T_Y|_{\mathcal{M}}$ and we can form a quotient E_1/E_0 which will be a *vector bundle stack* over $\mathcal{M}$.

So we obtain the embedding $C_{\mathcal{M}} \hookrightarrow N_{\mathcal{M}} \hookrightarrow E_1/E_0$ and the analogue of Proposition 4.6 holds for vector bundle stacks! So we can again define $[\mathcal{M}]^{vir} := 0^*(C_{\mathcal{M}})$, where 0 it the zero stction of the vector bundle (stack) $E_1/E_0 \to \mathcal{M}$.

The other way to do (even more simple) is the following: we can now form a fibre product $C := C_{\mathcal{M}} \times_{E_1/E_0} E_1$ and note that directly from the definitions C is now a scheme which is embedded into a vector bundle E_1. So we can define $[\mathcal{M}]^{vir} := 0^*(C)$ where $0\colon \mathcal{M} \hookrightarrow E_1$ is the zero section.