1. Artin algebras and deformation functors

Let $k$ be a field. Recall that a commutative algebra $A$ over $k$ is called Artinian if any descending chain of ideals stabilizes. An Artin algebra $A$ is called local if it is a local ring with residue field $k$, we denote by $m_A \subset A$ the maximal ideal of $A$.

Remark 1.1. Algebra $A$ is Artinian iff $\text{dim } A < \infty$ and $A$ is local.

Remark 1.2. Every Artin algebra is Noetherian.

Lemma 1.3. We have $m_A^N = 0$ for $N \gg 0$.

Proof. Indeed by Krull’s intersection theorem we have $\bigcap_{i \in \mathbb{Z}_{\geq 0}} m_A^i = 0$ and now the claim follows from the descending property applied to the ideals $m_A^i$. \hfill $\square$

We denote by $\text{Art}_k$ the category of local Artin $k$-algebras $A$ with residue field $k$.

The main example for us will be $A = k[\epsilon]/\epsilon^2$.

Definition 1.4. A deformation functor is a covariant functor $D: \text{Art}_k \to \text{Set}$ such that $D(k) = \{\ast\}$, where $\{\ast\}$ is some fixed one-element set.

Let us give some examples of deformation functors.

Definition 1.5. For any scheme $X$ we define a functor of points $h_X: \text{Art}_k \to \text{Set}$ given by $A \mapsto \text{Maps}(\text{Spec } A, X)$. If $X = \text{Spec } R$ then $h_R(A) = \text{Mor}(R, A)$.

Definition 1.6. Let $B \twoheadrightarrow A$ be a surjection with kernel $K$, $A, B \in \text{Art}_k$. We say

$$0 \to K \to B \to A \to 0$$

is a small extension if $m_B K = 0$.

The simplest example is $A = k$, $B = k[\epsilon]/\epsilon^2$, $K = (\epsilon)$.

Remark 1.7. Let $N$ be a module over $A$, we can associate to it the following ring: $A \ast N$, as a vector space we have $A \ast N := A \oplus N$ and the ring structure is the following: $(a_1, n_1)(a_2, n_2) = (a_1a_2, a_1n_2 + a_2n_1)$. Note that $N \subset A \ast N$ is an ideal such that $N^2 = 0$. Note also that $A \ast N = A \oplus \epsilon N/(\epsilon^2)$. Extensions $A \ast N$ are called trivial.

Lemma 1.8. Any surjection $B \twoheadrightarrow A$ can be obtained as a composition of small extensions.

Proof. Recall that $K \subset m_B$ is the kernel of our surjection $\pi: B \twoheadrightarrow A$. Note that by lemma 1.3 there exists $N \in \mathbb{Z}_{\geq 0}$ such that $m_B^N = 0$, hence, $K^N = 0$. Set $B_i := B/K^i$, $i = 1, \ldots, N$ and note that $B_0 = A$, $B_N = B$. It follows from the definitions that the natural surjections $B_i \twoheadrightarrow B_{i-1}$ define small extensions. Now $B \twoheadrightarrow A$ is the composition $B = B_N \twoheadrightarrow B_{N-1} \twoheadrightarrow \ldots \twoheadrightarrow B_1 \twoheadrightarrow B_0 = A$. \hfill $\square$
Lemma 1.9. There exists a surjection $k[[x_1, \ldots, x_d]] \to R$.

Proof. Recall that $R$ is complete so we have the isomorphism $R \xrightarrow{\sim} \varprojlim R/m_R^n$. It is enough to construct compatible surjections $k[[x_1, \ldots, x_d]] \to R/m_R^n$. This is an easy inductive construction. □

Consider now a small extension $B \to A$. Let us analyze the morphism $h_R(B) = \text{Mor}(R, B) \to \text{Mor}(R, A) = h_R(A)$.

1. Assume that $R = k[[x_1, \ldots, x_d]]$. The morphism $h_R(B) \to h_R(A)$ is surjective since if $f \in \text{Mor}(R, A)$ is some morphism and $[\tilde{x}_i] := f(x_i)$ then let $\tilde{x}_i \in B$ be any liftings of $[\tilde{x}_i]$ and we can define $\tilde{f} : R \to B$ by $\tilde{f}(x_i) = \tilde{x}_i$. It remain to describe the fibers of the morphism $h_R(B) \to h_R(A)$. Note that two functions $\tilde{f}_1, \tilde{f}_2 \in \text{Mor}(R, B)$ restrict to the same function $f \in \text{Mor}(R, A)$ iff the image of $\tilde{f}_1 - \tilde{f}_2$ lies in $K \subset B$. It is easy to deduce from $K^2 = 0$ that $\tilde{f}_1 - \tilde{f}_2 : R \to K$ is a derivation over $k$. Here the structure of $R$-module on $K$ comes from the homomorphism $f : R \to A$ and the natural action $A \rightharpoonup K$ (again use that $K^2 = 0$). To show this we just write

\[(\tilde{f}_1 - \tilde{f}_2)(ab) = \tilde{f}_1(a)\tilde{f}_1(b) - \tilde{f}_2(a)\tilde{f}_2(b) = \tilde{f}_1(a)\tilde{f}_2(b) + \tilde{f}_1(b)\tilde{f}_2(a) - 2\tilde{f}_2(a)\tilde{f}_2(b) = \]
\[= \tilde{f}_2(b)(\tilde{f}_1(a) - \tilde{f}_2(a)) + \tilde{f}_2(a)(\tilde{f}_1(b) - \tilde{f}_2(b)) = b \cdot (\tilde{f}_1 - \tilde{f}_2)(a) + a \cdot (\tilde{f}_1 - \tilde{f}_2)(b),\]

in the second equality we use that $(\tilde{f}_1 - \tilde{f}_2)(\tilde{f}_1 - \tilde{f}_2)(b) = 0$ since $K^2 = 0$. Note now that $\tilde{f}_1 - \tilde{f}_2$ is uniquely determined by its restriction to $m_S$ and moreover $(\tilde{f}_1 - \tilde{f}_2)(m_S^2) \subset m_RK = 0$ so we conclude that $\tilde{f}_1 - \tilde{f}_2$ is uniquely determined by its restriction to $m_R/m_R^2$. We obtain the following exact sequence

\[\text{Hom}(m_R/m_R^2, K) \to h_R(B) \to h_R(A) \to 0\]

Note that we have the identification $\text{Hom}(m_R/m_R^2, K) = (m_R/m_R^2)^* \otimes K$.

2. Consider now the case of general $R$. By Lemma (1.9) we have a surjection $\pi : S := k[[x_1, \ldots, x_d]] \to R$ which induces an isomorphism $m_S/m_S^2 \xrightarrow{\sim} m_R/m_R^2$. We set $I := \ker \pi$ and note that $I \subset m_S^2$. In the same way as above it is easy to see that the kernel of the map $\text{Mor}(R, B) \to \text{Mor}(R, A)$ identifies with $\text{Hom}(m_R/m_R^2, K) = (m_R/m_R)^* \otimes K$. We can now define some space and a morphism $\text{ob}$ such that $f \in h_R(A)$ has a lifting $\tilde{f}$ iff $\text{ob}(f) = 0$. Pick $f \in h_R(A)$ and consider the morphism $g := f \circ \pi$. Note that the morphism $\text{Mor}(S, B) \to \text{Mor}(S, A)$ is surjective. The lifting $\tilde{f}$ exists iff there exists a lifting $\tilde{g} \in \text{Mor}(S, B)$ such that $\tilde{g}|_I = 0$. Note now that for any two liftings $\tilde{g}_1, \tilde{g}_2 \in \text{Mor}(S, B)$ the morphism $\tilde{g}_1 - \tilde{g}_2$ is a derivative and $I \subset m_S^2$ so $\tilde{g}_1|_I = \tilde{g}_2|_I$. We conclude that the morphism $f \mapsto \tilde{g}|_I \in \text{Hom}(I/m_SI, K)$. So we obtain an exact sequence

\[\text{Hom}(m_R/m_R^2, K) \to h_R(B) \to h_R(A) \to \text{Hom}(I/m_SI, K)\]

Note that $m_R/m_R^2$ is the fiber at the point $m_R$ of the cotangent sheaf $\Omega_R$ and $(I/m_SI)^*$ is exactly the fiber of normal sheaf $N_{\mathcal{M}/Y}$, where $\mathcal{M} := \text{Spec } R$, $Y := \text{Spec } S$. 

Let $R$ be a complete local $k$-algebra such that $\dim_k(m_R/m_R^2) = d < \infty$. 

2 TANGENT-OBSTRUCTION COMPLEX TO MODULI PROBLEMS
Remark 1.10. In general if we have a closed embedding $M \hookrightarrow Y$ of some scheme into the affine scheme then obstructions for $M$ are $N_{M/Y}$ and deformations are controled by $\mathcal{T}_M$.

Note that there exists a natural morphism $\mathcal{T}_Y|_M \to N_{M/Y}$ which kernel is exactly $N_{M/Y}$. We have a morphism of complexes $[\mathcal{T}_M \to \mathcal{N}_{M/Y}] \to [\mathcal{T}_Y|_M \to N_{M/Y}]$ in degrees $-1, 0$ which is an isomorphism at the level of $-1$ and surjective at $0$ cohomology. This is a general property for obstruction theories to have a morphism to a so called tangent complex which can be represented in derived category by the complex $[\mathcal{T}_Y|_M \to N_{M/Y}]$ (note that cohomologies of $[\mathcal{T}_Y|_M \to N_{M/Y}]$ does not depend on the embedding of $M$ into $Y$).

1.1. Let us now discuss local structure in terms of tangent/obstruction complex.

Let us now discuss the relation between a local structure of a moduli functor $M: \text{Sch} \to \text{Set}$ at some point $w \in M(k)$ and its tangent-obstruction $T_wM \to \text{Ob}_wM$. There exists the natural homomorphism of algebras $\hat{S}^\bullet((\text{Ob}_wM)^*) \to \hat{S}^\bullet((T_wM)^*)$ called Kuranishi map. We will construct it later.

Remark 1.11. Recall that for a vector space $E$ over $k$ we denote by $\hat{S}^\bullet(E)$ the inverse limit $\bigoplus_{k=0}^\infty S^k(N)$.

Let us denote by $M^\wedge$ the formal neighbourhood of the point $w$ i.e. a functor $\text{Art}_k \to \text{Set}$ given by $A \mapsto M(A) \times_{M(A/m_A)} \{w\}$.

Remark 1.12. Here we use the isomorphism $k \cong A/m$ to obtain an identification $M(k) \cong M(A/m_A)$ so $w$ defines a point $w \in M(A/m_A)$ and $M^\wedge(A)$ is a fiber over $\{w\}$ of the natural morphism $M(A) \to M(A/m_A)$. Note that when $M$ is represented by a scheme $\text{Spec}R$ and $m \subset R$ is the maximal ideal corresponding to a point $w$ then $M^\wedge$ is represented by $R^\wedge := \lim \frac{R}{m^N}$. This follows from the fact that the homomorphism from $R^\wedge$ to $\text{Art}_k$ is the same as a homomorphism from $R$ to $A$ such that $m$ maps to $m_A$.

We assume that the functor $M$ is (locally) represented by a scheme $M$ and there is a tangent/obstruction theory on $M$ which induces on $M^\wedge$ our tangent/obstruction for $\text{Spec} \hat{O}_{M,w}$ (as above).

Proposition 1.13. The functor $M^\wedge$ is represented by the scheme $\text{Spec}(\hat{S}^\bullet(T_w^*)) \otimes \hat{S}^\bullet(\text{Ob}_w^*)$ k.

Proof. We can assume that $M$ is represented by some scheme $X$ of finite type. Note that the question is local so we can assume that $X = \text{Spec} R$, where $R$ is a complete local ring such that $\dim_k(m_R/m_R^2) = d < \infty$. Recall that in this case we have $T = \text{Hom}(m_R/m_R^2, K)$, $\text{Ob} = \text{Hom}(I/Im_S, K)$. Let us for simplicity assume that $K = k$ so we have $T^* = m_R/m_R^2$, $\text{Ob}^* = I/Im_S$. Then the Kuranishi map $\hat{S}^\bullet(I/Im_S) \to \hat{S}^\bullet(m_R/m_R^2) = S$ is induced by the natural embedding $I \hookrightarrow S$. Note now that $S \otimes \hat{S}^\bullet(I/Im_S) k$ is exactly $S/I \simeq R$ and the claim follows. Let us be more accurate here. Let $f_1, \ldots, f_n$
be generators of $I \subset S$. Let $[f_i] \in I/I_m$ be a basis in $I/I_m$ and $f_i \in I$ be any representatives. We claim that $f_i$ generate $I$ as a module over $S$. This is an immediate consequence of the Nakayama lemma. Now our morphism $K$ is induced by the morphism $I/I_m \rightarrow S$, $[f_i] \mapsto f_i$.

Let us now make an important comment. Recall that a functor $F$ is called formally smooth if the morphism $F(B) \rightarrow F(A)$ is surjective for any simple extension $0 \rightarrow K \rightarrow B \rightarrow A \rightarrow 0$. It is a standard fact that if $F$ is represented by a scheme $X$ of a finite type then the formal smoothness of $F$ is equivalent to the smoothness of $X$. Note now that formal smoothness implies that there are no obstructions so by Proposition 1.13 locally $F$ should be isomorphic to $T_w F$. This is exactly the case for smooth schemes. So we actually already know Proposition 1.13 for functors $F$ with no obstructions (formally smooth functors).

2. Obstructions for $G/H$

Recall that the fiber of Ob at the point $(C, p_1, \ldots, p_n, f) \in \mathcal{M}_{0,n,\beta}(X)$ is $H^1(C, f^*TX)$. Variety $X$ is called convex if Ob = 0. Goal of this section is to show that homogeneous spaces are convex (in particular, flag varieties) so the corresponding moduli spaces of stable maps are smooth.

**Proposition 2.1.** Let $G$ be an algebraic group and $H \subset G$ is a subgroup. Then for $X = G/H$ the vector bundle $TX$ is globally generated.

**Proof.** The action $G \acts X$ induces a morphism $\text{inf}: \mathfrak{g} \rightarrow \Gamma(X, k)$. It follows from the fact that the action of $G$ is transitive that the image of $\text{inf}$ generates $TX$. □

**Proposition 2.2.** Assume that $X$ is such that $TX$ is globally generated. Then $X$ is convex.

**Proof.** Consider some morphism $\mu: C \rightarrow X$. Our goal is to show that $H^1(f^*TX, C) = 0$. Note that $f^*TX$ is globally generated so we have a short exact sequence

$$0 \rightarrow K \rightarrow \Gamma(f^*TX, C) \otimes \mathcal{O}_C \rightarrow f^*TX \rightarrow 0$$

which induces the long exact sequence

$$\rightarrow \Gamma(f^*TX, C) \otimes H^1(C, \mathcal{O}_C) \rightarrow H^1(C, f^*TX) \rightarrow H^2(C, K) \rightarrow .$$

Note now that $H^2(C, K) = 0$ since $\dim C = 1$ and $H^1(C, \mathcal{O}_C) = H^0(C, \omega_C) = H^0(C, \mathcal{O}_C(-2)) = 0$. We conclude that $H^1(C, f^*TX) = 0$. □

**Remark 2.3.** Note that every irreducible component of $C$ is smooth (it is isomorphic to $\mathbb{P}^1$) so the dualizing complex of $C$ is $\omega_C[1] = \mathcal{O}_C(-2)[1]$.

3. Main example of section of a vector bundle

Starting from a variety $M$ together with a closed embedding $\iota: M \hookrightarrow \mathbb{A}^m$ given by some functions $(f_1, \ldots, f_n)$ we can associate to it the following complex of sheaves on $M$:

$$\mathcal{I}_{\mathbb{A}^m} |_M \rightarrow \mathcal{O}_M^{\oplus n},$$
where \( \sigma \) is induced by \( dt \). Let us now give the algebraic description of the morphism \( \sigma \). We give the description of the dual morphism \( \sigma^* : \mathcal{O}_{M}^{\oplus n} \to \mathcal{T}_{M}^{\gamma} \). Let \( S \) be the ring of global functions on \( \mathbb{A}^n \) and \( R \) be the ring of global functions on \( X \). We have the natural surjection \( S \to R \) with kernel \( I \) generated by \( f_1, \ldots, f_n \). Note that \( \mathcal{T}_{M}^{\gamma} = R \otimes S \Omega_S \). It follows from the definitions that the morphism \( \sigma^* : R^{\oplus n} \to R \otimes S \Omega_S \) is induced by the composition \( R^{\oplus n} \to I \to R \otimes S \Omega_S \) where the first morphism sends \((0, \ldots, 1, \ldots, 0)\) to \( f_i \) and the second morphism is given by \( f \mapsto 1 \otimes df \).

**Remark 3.1.** In slightly more general situation when we have a closed embedding of \( M \hookrightarrow Y \) as zeroes of some section of some vector bundle \( E \to Y \) then we always have a morphism \( TY|_M \to E|_M \) induced by the composition \( TY|_M \to N_{M/Y} \hookrightarrow E|_M \).

Fix now a point \( x \in M \). Our goal is to describe obstructions at this point. So we can assume that \( R, S \) are local and complete with maximal ideals corresponding to \( x \). We claim that the obstructions lie in \( O_x := \operatorname{coker} \sigma_x \). Consider the morphism \( F : k^n \to S = S^*(m_S/m_S^2) \) given by \((0, \ldots, 1, \ldots, 0) \mapsto f_i \). Note now that the induced morphism \( k^n \to m_S/m_S^2 \) coincides with \( \sigma_x^* \) after the identification \((R \otimes_S \Omega_S)_x = k \otimes_R (R \otimes_S \Omega_S) = k \otimes_S \Omega_S = m_S/m_S^2 \). Note also that the image of the morphism \( F \) generates an ideal \( I \).

Let us now start from a simple extension of Artin rings

\[
0 \to K \to B \to A \to 0
\]

and consider a morphism \( \varphi : S/(F) \to A \). We want extend it to a morphism \( S/(F) \to B \). Note that we always have an extension \( g : S \to B \) of the composition \( S \to S/(F) \to A \).

It follows from the definitions that the morphism \( k^{\oplus n} \xrightarrow{F} S \xrightarrow{g} B \) factors through \( \sigma : k^{\oplus n} \to K \). Let \( \text{ob}(\varphi, A, B) \) be the image of \( \sigma \) under the obvious morphism \( k^{\oplus n} \otimes K \to O \otimes K \). Indeed the condition that \( \text{ob}(\varphi, A, B) \) is equivalent to the fact that the fact that \( \sigma : k^{\oplus n} \to K \) lifts to the morphism \( h : m_S/m_S^2 \to K \) and we denote by \( \hat{h} : S \to B \) the corresponding homomorphism of algebras. It remains to note that \( g - \hat{h} \) factors through \( R = S/(F) \to B \) extending the morphism \( \varphi \).

**Remark 3.2.** The fact that \( (g - \hat{h})(F) = 0 \) is equivalent to \( g \circ F = \hat{h} \circ F \) but \( g \circ F = o \) which lifts to \( h \) and the claim follows. The fact that \( g - \hat{h} \) extends \( \varphi \) follows from the definition of \( g \) and the fact that \( h \) maps to \( K \).

So we see that for this exampe the tangent-obstruction complex is homologies of the complex \( \mathcal{T}_Y|_M \to \mathcal{E}|_M \). Such obstruction theories are called perfect (when they from a complex of locally trivial sheaves) and in these cases one can define virtual fundamental class of \( M \) (in dimension \( \dim \mathcal{E} = \dim \mathcal{E}^1 \) where \( \mathcal{E}^0 \to \mathcal{E}^1 \) is our obstruction theory).

Note also that we have a natural morphism \([\mathcal{T}_Y|_M \to N_{M/Y}] \to [\mathcal{T}_Y|_M \to \mathcal{E}|_M]\) induced by the morphisms \( \mathcal{T}_Y|_M \xrightarrow{\text{Id}} \mathcal{T}_Y|_M, N_{M/Y} \xrightarrow{\text{Id}} \mathcal{E}|_M \) and this morphism is an isomorphism on 0 cohomologies and is injective on 1st cohomologies (people standatdly consider dual complexes).
4. Virtual fundamental complex in our main example

Let $\mathcal{M}$ be the moduli functor of virtual dimension $n$ (virtual dimension depends on the (perfect) tangent/obstruction theory on $\mathcal{M}$). Our goal is to define a fundamental class of $\mathcal{M}$ which will be an element of $A_n(\mathcal{M})$ (the $n$-th Chow group). There is a homomorphism $A_n(\mathcal{M}) \to H_{2n}^{BM}(\mathcal{M}, \mathbb{C})$ so one can think about our fundamental class as about an element of Borel-Moore homology.

**Remark 4.1.** Let us recall the definition of Chow groups of scheme $\mathcal{M}$. This is a quotient of the group (over $\mathbb{C}$) generated by subvarieties of $\mathcal{M}$ by rational equivalences i.e. if we have a subvariety $W \subset \mathcal{M}$ of dimension $i + 1$ and a rational function $f$ on $W$ then the divisor $(f)$ of $f$ should be equal to zero in $A_i(\mathcal{M})$.

**Remark 4.2.** We recall that if $\mathcal{M}$ is a topological space then the Borel-Moore homology $H_{\bullet}^{BM}(\mathcal{M}, \mathbb{C})$ can be defined as $H_{\bullet}(\overline{\mathcal{M}}, \mathbb{M} \setminus \mathcal{M})$, where $\overline{\mathcal{M}}$ is any compactification of $\mathcal{M}$ (for example, one-point compactification).

**Remark 4.3.** Let us finally remark that for (possibly noncompact) manifold $\mathcal{M}$ there exists an isomorphism $H_{2n}^{BM}(\mathcal{M}, \mathbb{C}) \simeq H^{\dim \mathcal{M}-i}(\mathcal{M}, \mathbb{C})$.

**4.1. Notations/definitions.** Let us recall come notations. Let $\mathcal{M} \hookrightarrow Y$ be a closed embedding of schemes and $\mathcal{J}$ is the ideal defining $\mathcal{M}$. Then we define $N_{\mathcal{M}/Y} := (\mathcal{J}/\mathcal{J}^2)\wedge$, this is a sheaf on $\mathcal{M}$. We denote by $N_{\mathcal{M}/Y} \to \mathcal{M}$ the corresponding total space. Then any morphism $E \to \mathcal{M}$ induces a morphism $P \to \mathcal{M}$ which is defined as the relative spectrum $\text{Spec} \, S_{\mathcal{O}_X}^{\bullet}(\mathcal{J}/\mathcal{J}^2)$.

**Remark 4.4.** There is a general way to pass from a coherent sheaf $\mathcal{E}$ on a scheme $X$ to the corresponding total space $E \to X$ over $X$. Assume first that $X = \text{Spec} \, A$ is an affine scheme. Then we have $\mathcal{E} = \mathcal{M}$ for some $A$-module $M$ ($M = \Gamma(X, \mathcal{E})$). Now we can define $E := \text{Spec} \, S_x^{\bullet}(M)$, here $S_x^{\bullet}(M) := \bigoplus_{i \geq 0} S_x^i(M)$ and $S_x^i(M)$ is the quotient of the tensor product $M \otimes_A M \otimes_A \ldots \otimes_A M$ and $S_x^0(M) = A$. Note that we have the natural morphism $E \to X$, which corresponds to the embedding $A = S^0(M) \to S_x^i(M)$.

We can apply the same construction to the sheaf $\mathcal{E}$ (using some affine covering of $X$) and get the desired scheme $E$. Note that if $\mathcal{E}$ is locally trivial then $E$ is the total space of the vector bundle corresponding to $\mathcal{E} \wedge$. Note also that the construction is functorial in the following sense. If $\mathcal{E}$, $\mathcal{P}$ are coherent sheaves on $X$ and $E \to X$, $P \to X$ are the corresponding total spaces. Then any morphism $\mathcal{E} \to \mathcal{P}$ over $\mathcal{O}_X$ induces a morphism $P \to E$ of schemes over $X$ (clear from the construction).

In more details: morphism $\mathcal{E} \to \mathcal{P}$ induces a morphism the morphism $S_{\mathcal{O}_X}^{\bullet}(E) \to S_{\mathcal{O}_X}^{\bullet}(P)$ and so a morphism $P = \text{Spec}_{\mathcal{O}_X} S_{\mathcal{O}_X}^{\bullet}(\mathcal{P}) \to \text{Spec}_{\mathcal{O}_X} S_{\mathcal{O}_X}^{\bullet}(\mathcal{E}) = E$.

We have the natural morphism of sheaves $\mathcal{J}/\mathcal{J}^2 \to \Omega_Y|_M$ which induces the morphism $TY|_M \to N_{\mathcal{M}/Y}$. Note that when the embedding $\mathcal{M} \hookrightarrow Y$ is regular then the sheaf $\mathcal{J}/\mathcal{J}^2$ is locally trivial and we have the identification $\mathcal{J}/\mathcal{J}^2 \to (\mathcal{J}/\mathcal{J}^2)^{\wedge}$ so $N_{\mathcal{M}/Y}$ is exactly the normal bundle to $\mathcal{M} \subset Y$.

**Remark 4.5.** Note that we have a short exact sequence of sheaves on $\mathcal{M}$

$$0 \to \mathcal{T}_M \to \mathcal{T}_Y|_M \to N_{\mathcal{M}/Y}$$
and the morphism $\mathcal{I}_Y|_M \to N_{M/Y}$ is surjective when $M$ is smooth. In general we can define $\mathcal{I}_M$ to be the cokernel of this morphism. This sheaf is called a tangent sheaf and it actually does not depend on the embedding $M \hookrightarrow Y$. So in general we have the following exact sequence of sheaves

$$0 \to \mathcal{I}_M \to \mathcal{I}_Y|_M \to N_{M/Y} \to \mathcal{I}_M^1 \to 0.$$ 

### 4.2. Simple simple case.

We start from the simplest case. Let $E$ be a vector bundle of rank $r$ on a smooth variety $A$ of dimension $n$ and $s \in \Gamma(A, E)$. Assume that $M = Z(s)$ the zero locus of $s$.

Recall that if $P \to M$ is a (real) vector bundle of rank $k$ over a (smooth) variety $M$ then we have a canonically defined Euler class $eu(P) \in H^k(M, \mathbb{C})$ such that the Poincare dual class $PD_M(eu(P)) \in H_{\dim M - k}(M, \mathbb{C})$ is the class of zeroes of a generic section of $P$.

Let us return to our example. Recall that $\text{vdim} M = \dim Y - \text{rk} E$. The simplest case is when $s$ is generic so $M$ is smooth and $\dim M = \text{vdim} M$. Then we already know that we must have $[M]^{\text{vir}} := [M]$ and the last can be thought as $PD_M(\text{eu}(0))$. Note that in terms of $\sigma: TY|_M \to E|_M$ this happens exactly when $\sigma$ is surjective (i.e. $\text{Ob} = 0$).

Assume now that $M$ is still smooth but has the wrong dimension (for example $s = 0$ and $M = Y$). This is the case when section $s$ takes values in some vector subbundle $E' \subset E$ of rank $r'$ and is transverse to the zero section of $E'$ and (in $C^\infty$-setting) we have a decomposition $E = E' \oplus (E/E')$. We see that $s = (s', 0)$ for some $s' \in \Gamma(A, E')$. Note that we can now deform $(s', 0)$ to a section $(s', \epsilon)$, $\epsilon \in \Gamma(A, E/E')$ with $\epsilon$ transverse to the zero section of $E/E'$ and we see that our virtual class should be a class of generic section on the bundle $E/E'$ i.e. can be defined as follows $[M]^{\text{vir}} := PD_M(\text{eu}(E/E'|_M))$.

Note now that the sheaf $(E/E'|_M)|_M$ canonically identifies with $coker \sigma = \text{Ob}$. So we conclude that in this case we define $[M]^{\text{vir}} := PD_M(\text{eu}(\text{Ob}))$. This suggests us the following definition when $\text{Ob}$ is a vector bundle over $M$.

$$[M]^{\text{vir}} := PD_M(\text{eu}(\text{Ob})).$$

Let us check that this class (in our example) will lie in the correct dimension (i.e. in dimension $\text{vdim} M = \dim Y - \text{rk} E$). Indeed we have an exact sequence

$$0 \to TM \to TY|_M \to E \to Ob \to 0$$

so $\text{rk} Ob = \text{rk} E + \dim M - \dim Y$ i.e. $\text{eu}(\text{Ob}) \in H^{2 \text{rk} E + 2 \dim M - 2 \dim Y}(M, \mathbb{C})$. It follows that $PD(\text{eu}(\text{Ob})) \in H_{2 \dim Y - 2 \text{rk} E}(M, \mathbb{C})$ and that is it.

Recall that we want to intersect zero section of $E \to Y$ with a transverse section of $E$. Topologically we know that the answer is always $PD_Y(eu E)$ but the problem is that this is a class in cohomologies of $Y$ not of $M$. So in general we want to construct some element $[M]^{\text{vir}} \in H_{2 \text{vdim} M}(M, \mathbb{C})$ such that its push-forward to $H_{2 \text{vdim} M}(M, \mathbb{C})$ is $PD_Y(eu E)$. So we want to deform $M$ inside itself. The idea is not to deform $M$ inside $Y$ but to deform $Y$ itself (to a certain cone) such that this deformation is trivial being restricted to $M \hookrightarrow Y$. There is a canonical construction of such a deformation called a deformation to normal cone.
4.3. Rees construction and normal cone. Let us now discuss some important general construction (called Rees construction). Let $A$ be a commutative algebra over $\mathbb{C}$. Assume that we are given an increasing filtration $\ldots \subset A_{-1} \subset A_0 \subset A_1 \subset \ldots$ on the algebra $A$ such that $A_i \cdot A_j \subset A_{i+j}$. We can form the following algebra $A_h$ over $\mathbb{C}[h]$, $A_h := \bigoplus h^i A_i$ and note that this is indeed an algebra with respect to the multiplication $a_i h^i \cdot a_j h^j = a_i a_j h^{i+j}$ and $h^i \in A_h$ for any $i \geq 0$ since $1 \in A_0 \subset A_1$. We obtain the family $\pi: \text{Spec } A_h \to \text{Spec}(\mathbb{C}[h])$. Let us first of all note that $A_h$ is $h$-graded so the morphism is actually $\mathbb{C}^\times$-equivariant with respect to the standard action $\mathbb{C}^\times \acts A_h$. We conclude that for any $t \neq 0$ the fiber $\pi^{-1}(t)$ identifies with $\pi^{-1}(1)$. Let us compute $\pi^{-1}(1)$.

By the definition it is the spectrum of the quotient $\bigoplus_i h^i A_i / (h - 1)$ i.e. equals to $\bigoplus_i A_i = A$. Let us now compute the fiber $\pi^{-1}(0)$. By the definition it is the spectrum of the quotient $\bigoplus_i h^i A_i / (h)$ and it is easy to see that this algebra naturally identifies with $\bigoplus_i A_i / A_{i+1}$. We conclude that the family $\pi$ deforms Spec $A$ to Spec $A_h$.

Starting from a closed embedding $M \hookrightarrow Y$ with defining ideal sheaf $I$ we can define a filtration in $\mathcal{O}_Y$ by $\mathcal{I}$, where $\mathcal{I} = \mathcal{O}_Y$ for $i \leq 0$. In other words we should consider
\[
\ldots \oplus h^{-2} \mathcal{J}^2 \oplus h^{-1} \mathcal{J} \oplus \mathcal{O}_Y \oplus h\mathcal{O}_Y \oplus \ldots.
\]
So we obtain the family $\tilde{Y} \to \mathbb{A}^1$ such that the fiber over $t \neq 0$ is isomorphic to $Y$ and the fiber over $0$ is $C_{M/Y} := \text{Spec}(\bigoplus_{i \geq 0} \mathcal{I}^i / \mathcal{I}^{i+1})$. Note also that we have the embedding $M \times \mathbb{A}^1 \hookrightarrow \tilde{Y}$ corresponding to the ideal $\bigoplus_i h^i \mathcal{J}^i$ (the quotient by this ideal is naturally $\mathcal{O}_{M[h]}$).

So we see that $Y$ naturally deforms to $C_{M/Y}$ together with the embedding $M \hookrightarrow Y$. It is natural to assume that intersection does not depend on the deformation so we can pass from the embedding $M \hookrightarrow Y$ to the embedding $M \hookrightarrow C_{M/Y}$.

4.4. General case of simple case. Recall that we denote by $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of $\mathcal{M}$ and $C_{M/Y} := \text{Spec}(\bigoplus_{i=0}^\infty \mathcal{I}^i / \mathcal{I}^{i+1})$ and note that we have the natural surjection $\bigoplus_{i=0}^\infty \mathcal{J}^i / \mathcal{J}^{i+1} \to \bigoplus_{i=0}^\infty \mathcal{I}^i / \mathcal{I}^{i+1}$ which induces a closed embedding embedding $C_{M/Y} \hookrightarrow N_{M/Y}$. We also have the natural embedding $N_{M/Y} \hookrightarrow E_M$ (induced by $s$). Let us now recall how the morphism $N_{M/Y} \to E_M$ is constructed.

Recall that $N_{M/Y} = (\mathcal{J} / \mathcal{J}^2)^\vee$ and $s$ defines a morphism of sheaves $\mathcal{O}_Y \to \mathcal{E}$ and the dual morphism $s^\vee: \mathcal{E}^\vee \to \mathcal{O}_Y$. We claim that the image of this morphism is the ideal sheaf $\mathcal{J}$. It is enough to check this locally: when $Y = \text{Spec } S$ and $M = \text{Spec } R$ are both affine, $\mathcal{E} = \mathcal{O}_Y^{\oplus r}$ and $s$ is then given by $r$ functions $f_i \in S$ and the corresponding morphism $S \to S^{\oplus r}$ sends $s$ to $(sf_1, \ldots, sf_2)$ so the dual morphism $S^{\oplus r} \to S$ sends $(s_1, \ldots, s_r)$ to $f_1 s_1 + \ldots + f_r s_r$ and its image is exactly $I = (f_1, \ldots, f_r)$ the ideal of $M \hookrightarrow Y$.

So we obtain a morphism $\mathcal{E}^\vee \to \mathcal{J}$ which induces a morphism $\mathcal{E}^\vee_M \to \mathcal{J} / \mathcal{J}^2$ of sheaves on $M$. Note that this morphism is surjective! This can be checked locally using the fact that $f_i$ generate the ideal $I$ over $S$ so $[f_i] \in I / I^2$ generate $I / I^2$ over $R = S/I$.

The surjection $\mathcal{E}^\vee_M \to \mathcal{J} / \mathcal{J}^2$ induces the desired embedding $N_{M/Y} \hookrightarrow E_M$. One should note that the morphism $TY_M \to E$ that we have discussed at the previous lecture can be described as follows. It is induced by the morphism of sheaves $\mathcal{E}^\vee \to \mathcal{O}_Y |_M$ that is obtained as a composition of the morphisms $\mathcal{E}^\vee \to \mathcal{J} / \mathcal{J}^2 \to \mathcal{O}_Y |_M$ so (our
fundamental map which controls deformations/obstructions) \( TY|_M \to E|_M \) is exactly the composition \( TY|_M \to N_{M/Y} \hookrightarrow E|_M \).

4.4.1. Completing the construction. So we are in the following situation: We have an embedding \( C_{M/Y} \hookrightarrow E|_M \) and \( E|_M \to M \) is the vector bundle over \( M \). We can finally define \( [M]^{vir} := 0^*(C_{M/Y}) \) and we explain in the next section what it means.

Note now that in the case when \( s \) actually corresponds to some section \((s', 0)\)

4.5. Intersection with zero section in vector bundle. Intersection theory for normal cones was developed in Fulton’s book and works as follows.

Let us recall the following proposition.

**Proposition 4.6.** Let \( E \) be a rank \( r \) vector bundle on \( X \), \( p: E \to X \) then the pull-back homomorphism \( p^*: A_k(X) \to A_{k+r}(E) \) is an isomorphism, here \( A \) corresponds to Chow group.

**Remark 4.7.** Let us make a comment how to prove the surjectivity of the morphism \( p^* \). We use the induction on the dimension of \( X \). We can always assume that \( X \) is irreducible so there exists an open subset \( U \hookrightarrow X \) such that \( E|_U \) is trivial (so \( p^*|_U \) is an isomorphism). Set \( Z := X \setminus U \) and denote by \( \iota: Z \hookrightarrow X \) the closed embedding and by \( j: U \hookrightarrow X \) the open embedding. We have the following exact sequence

\[
A_k(Z) \xrightarrow{\iota^*} A_k(X) \xrightarrow{\iota^*} A_k(U) \to 0
\]

and the same exact sequence for \( E|_Z, E, E|_U \). Now from the surjectivity for \( E|_Z \) and \( E|_U \) the surjectivity of \( p^* \) follows.

Let \( s: X \hookrightarrow E \) be a zero section and \( \alpha \in A_k(E) \) then we can define \( s^*(\alpha) \) as the unique cycle in \( A_{k-r} \) such that \( p^*s^*(\alpha) = \alpha \). So we can now define intersections of closed subvarieties \( Z \subset E \) and zero section (of \( E \))! This is exactly what we need.

5. Virtual fundamental class for perfect tangent-obstruction theory

Let us now briefly describe the general approach to constructing of \( [M]^{vir} \) starting from a perfect tangent-obstruction theory \( T \).

Let us now define perfect obstruction theory on a scheme (more generally Deligne-Mumford stack) \( X \).

It consists of a complex \( \mathcal{E}^{-1} \to \mathcal{E}^0 \) of locally trivial sheaves on \( X \) such that for any embedding \( X \hookrightarrow Y \) into a smooth \( Y \) we have a morphism of complexes \( \mathcal{E}^{-1} \to \mathcal{E}^0 \) to \([N_{M/Y}^\vee \to \Omega_Y|_M]\) which is an isomorphism on \( h^0 \) and surjection on \( h^{-1} \).

**Remark 5.1.** Note that we have the following exact sequence

\[
0 \to (\mathcal{T}_M^1)^\vee \to N_{M/Y}^\vee \to \Omega_Y|_M \to \Omega_M \to 0
\]

so the cohomologies of the complex \([N_{M/Y}^\vee \to \Omega_Y|_M]\) do not depend on \( Y \). So we can actually consider \([N_{M/Y}^\vee \to \Omega_Y|_M]\) as an object of the derived category \( D^b(M) \) (this is a so-called truncated cotangent complex) and define perfect obstruction theory without \( Y \).
In general we construct a vector bundle \( V \) on \( M \) and a cone \( C \subset V \) and define \( [M]^\text{vir} := s^*(C) \), where \( s: M \hookrightarrow V \) is the zero section. Pair \( (C \subset V) \) is constructed purely in terms of tangent-obstruction complex of \( M \). Let us remark that the virtual fundamental class \( \text{depends} \) on the choice of the tangent-obstruction complex.

Let us roughly describe the construction of \( C \) and \( V \). Again choose an embedding \( M \hookrightarrow Y \), where \( Y \) is smooth (there are no vector bundle \( E \) now). We still have the canonical morphisms \( I^k/I^{k+1} \to S^k(\Omega_Y|_M) \) which induces a morphism \( T_Y|_M \to C_M/Y \) and we define \( C_M := C_{M/Y}/T_Y|_M \) (this is a stack quotient!). In the same way we can define the quotient \( N_M/T_Y|_M \) and note that we have the natural morphism \( C_M \to N_M \).

Recall also that we have our perfect obstruction theory \( \mathcal{E}^{-1} \to \mathcal{E}^0 \) which has a morphism to \( N_{M/Y}^\vee \to \Omega_Y|_M \). They induce morphisms of the corresponding schemes \( E_1, E_0, N_{M/Y}, T_Y|_M \) and we can form a quotient \( E_1/E_0 \) which will be a vector bundle stack over \( M \).

So we obtain the embedding \( C_M \hookrightarrow N_M \hookrightarrow E_1/E_0 \) and the analogue of Proposition 4.6 holds for vector bundle stacks! So we can again define \( [M]^\text{vir} := 0^*(C_M) \), where \( 0: M \hookrightarrow E_1 \) is the zero section.

The other way to do (even more simple) is the following: we can now form a fibre product \( C := C_M \times_{E_1/E_0} E_1 \) and note that directly from the definitions \( C \) is now a scheme which is embedded into a vector bundle \( E_1 \). So we can define \( [M]^\text{vir} := 0^*(C) \) where \( 0: M \hookrightarrow E_1 \) is the zero section.