CHARACTERS UP TO HOMOTOPY OF DERIVED ALGEBRAIC GROUPS

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ABSTRACT. For a Hopf DG-algebra corresponding to a derived algebraic group, we compute the homotopy limit of the associated cosimplicial system of DG-algebras given by the classifying space construction. The homotopy limit is taken in the model category of DG-categories. The objects of the resulting DG-category are Maurer-Cartan elements of Cobar(A), or 1-dimensional A_∞-comodules over A. These can be viewed as characters up to homotopy of the corresponding group.

1. Introduction

This is a report on a joint project in progress with S. Arkhipov.

In [BHW] and [AØ] the authors construct an explicit model for homotopy limit of a cosimplicial DG category as a derived totalization. We apply this construction to a cosimplicial system of DG algebras arising from a commutative Hopf DG-algebra A.

The objects of resulting DG category are shown to be the Maurer-Cartan elements of Cobar(A). If we think of A as of functions on the derived affine group, then our category is closely related to the category of representations up to homotopy introduced in [AC], the difference being that we allow our group to be derived but restrict ourselves to characters.

In parallel with [ACD], we study the monoidal structure in the resulting category.

2. Cosimplicial system of DG-algebras

Let $(A, m, 1, \Delta, \epsilon)$ be a (unital, counital) commutative Hopf DG-algebra. Consider its cosimplicial system of DG-algebras

$$k \rightrightarrows A \Rrightarrow A^{\otimes 2} \dots \tag{1}$$

Let d_n^i denote the face map $A^{\otimes n} \to A^{\otimes n+1}$ and s_n^i denote the degeneracy map $A^{\otimes n} \to A^{\otimes n-1}$. Then in the system above

$$d_n^i = \begin{cases} 1 \otimes id^{\otimes n} & i = 0 \\ id^{\otimes i-1} \otimes \Delta \otimes id^{\otimes n-i} & 0 < i < n \\ id^{\otimes n} \otimes 1 & i = n \end{cases}$$

$$s_n^i = \begin{cases} \epsilon \otimes id^{\otimes n-1} & i = 0 \\ id^{\otimes i-1} \otimes m \otimes id^{\otimes n-1} & 0 < i < n \\ id^{\otimes n-1} \otimes \epsilon & i = n \end{cases}$$

We view this as a cosimplicial DG category, and we want to obtain an explicit description of its homotopy limit with respect to Tabuada's model structure inverting quasiequivalences (see [T]). Denote this homotopy limit by $\mathfrak{A}$.

Applying Proposition 4.0.2 from [AØ] to (1), we obtain the following data.

Theorem 1. An object a in $\mathfrak{A}$ is an infinite sequence $\{a_i\}_{i\geq 1}$ with $a_i \in (A^{\otimes i})^{1-i}$ and a_1 homotopy invertible, subject to

$$\begin{aligned} d(a_2) &= a_1 \otimes a_1 - \Delta(a_1) \\ d(a_3) &= a_1 \otimes a_2 - a_2 \otimes a_1 - (1 \otimes \Delta + \Delta \otimes 1)(a_2) \\ &\dots \\ d(a_n) &= \\ \sum_{i=1}^{n-1} (-1)^{i+1} a_i \otimes a_{n-1} &- \sum_{i=0}^{n-2} (-1)^{i(n-1)} (1^{\otimes i} \otimes \Delta \otimes 1^{\otimes n-1-i})(a_{n-1}) \end{aligned} \tag{2}$$

A morphism $f\colon a \to b$ of degree m is also an infinite sequence $\{f_i\}_{i\geq 0}$ with $f_i \in (A^{\otimes i})^{-i}$, with differential given by

$$d(f)_i = d(f_i) + \sum (-1)^j (a_j \otimes f_{j-i} + f_j \otimes b_{j-i})$$

and composition given by

$$(g \circ f)_i = \sum g_j \otimes f_{i-j}.$$

3. Maurer-Cartan elements

For any (not necessarily counital or coaugmented) coalgebra C, recall the following definition of its Cobar construction.

Definition 2. As a graded space,

$$\mathrm{Cobar}(C) = \widehat{T}(A[-1]) = \prod_{i=0}^{\infty} C[-1]^{\otimes i}$$

The multiplication is that of a complete tensor algebra. The differential is given by $d = d_C + \Delta$ on generators and extends to the rest of the algebra by Leinbiz rule.

The following observation then can be made.

Proposition 3. The objects of $\mathfrak{A}$ are exactly the Maurer-Cartan elements of this DG-algebra, with one extra condition that their first component is homotopy invertible.

Note that in any DG-algebra A a Maurer-Cartan element c allows to twist the differential:

$$d_c(a) = d(a) + [c, a]$$

Denote the new algebra by ${}_cC_c$. For two Maurer-Cartan elements c_1 and c_2, denote by ${}_{c_1}C_{c_2}$ a complex obtained by considering A with the differential

$$d_{(c_1,c_2)}(a) = d(a) + c_1 a + a c_2.$$

This will not be a DG-algebra anymore (for the lack of multiplication satisfying the Leibniz rule), but it will be a ${}_{c_1}C_{c_1}$-${}_{c_2}C_{c_2}$ DG-bimodule.

Proposition 4. In $\mathfrak{A}$,

$$\mathfrak{A}(a, b) =_a \mathrm{Cobar}(A)_b.$$

So as a set, every $\mathfrak{A}(a, b)$ is isomorphic to $\mathrm{Cobar}(A)$; the composition $\mathfrak{A}(a, b) \otimes \mathfrak{A}(b, c) \to \mathfrak{A}(a, c)$ can now be interpreted simply as the multiplication in $\mathrm{Cobar}(A)$. Strictly isomorphic objects of $\mathfrak{A}$ correspond to gauge-equivalent Maurer-Cartan elements (recall that a and b are gauge-equivalent if $b = faf^{-1} + fd(f^{-1})$ for some invertible f of degree 0).

In $\mathrm{Cobar}(A)$, there is a distinguished Maurer-Cartan element, namely, $(0, 1, 0, 0, \ldots)$. Denote the corresponding object of $\mathfrak{A}$ by $\mathbb{I}$. Its endomorphisms are the well-known *reduced Cobar construction* $\mathrm{Cobar}_r(A)$ which exists for coaugmented DG-coalgebras; this is also the total cochain complex associated to the cosimplicial system (1).

4. Co-Morita equivalence

For any DG algebra A and Maurer-Cartan elements a, b it holds that

$${}_aA_b \otimes_{{}_bA_b} {}_bA_a =_a A_a,$$

so on the nose ${}_aA_b$ and ${}_bA_a$ are inverse bimodules. This gives an expectation for a Morita equivalence between $\mathfrak{A}$ and $\mathrm{Cobar}_r A = \mathrm{End}_{\mathfrak{A}}(\mathbb{I})$. However, sometimes these bimodules may be acyclic, and derived tensoring by an acyclic bimodule cannot induce an equivalence of derived categories. To make things work one might consider not derived categories but instead Positselski's coderived categories (see [P]), where the class of acyclic objects is replaced by a smaller class of coacyclic objects.

Definition 5. For a DG-algebra A, the subcategory $Coacycl \subset Ho(A)$ is the smallest triangulated subcategory containing totalizations of exact triples of modules and closed with respect to infinite direct sums.

Definition 6. The coderived category is $D^{co}(A) = Ho(A)/Coacycl$.

We expect, for any DG algebra A, to have an equivalence of coderived categories

$$\otimes_b A_a \colon D^{co}({}_aA_a) \simeq D^{co}({}_bA_b)$$

.

5. A_∞-comodules and homotopy characters

Another interpretation of $\mathfrak{A}$ is via A_∞-comodules.

Proposition 7. An object of $\mathfrak{A}$ is an A_∞-comodule structure over A on k. The Hom-complexes are the complexes of (non-strict) A_∞-comodule morphisms.

Note that if A was a coalgebra of functions on some group, then comodules over this coalgebra would correspond to representations of the group. [AC] extend this by saying that A_∞-comodules correspond to representations up to homotopy. We follow this intuition and view the objects of $\mathfrak{A}$ as homotopy characters of a derived affine group corresponding to A.

6. Example of an algebraic group

In the case when A is a Hopf algebra of functions on a group (concentrated in degree 0), the category $\mathfrak{A}$ will have honest characters as objects, and Hom-complexes will compute Exts between them.

Example 8. Let G be the group of invertible upper triangular 2×2 matrices over $\mathbb{C}$. Consider the following functions:

$$x\begin{pmatrix} a & c \\ 0 & b\end{pmatrix} = a; \quad y\begin{pmatrix} a & c \\ 0 & b\end{pmatrix} = b; \quad z\begin{pmatrix} a & c \\ 0 & b\end{pmatrix} = c.$$

The Hopf algebra of regular functions on G is $\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z]$, with coproduct

$$\Delta(x^{\pm 1}) = x^{\pm 1} \otimes x^{\pm 1};$$
$$\Delta(y^{\pm 1}) = y^{\pm 1} \otimes y^{\pm 1};$$
$$\Delta(z) = x \otimes z + z \otimes y.$$

Over G we have $Ext^1(1, xy^{-1}) = \mathbb{C}$. In our Holim category, the Hom complex between 1 and xy^{-1} is

$$\mathbb{C} \longrightarrow \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z] \longrightarrow \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z]^{\otimes 2} \longrightarrow \dots$$

where the first differential is multiplication by $1 - xy^{-1}$, and the second differential is given by $d(f) = f \otimes 1 + xy^{-1} \otimes f + \Delta(f)$. The kernel of it is generated by $1 - xy^{-1}$ and $y^{-1}z$, the latter being a representative for the nontrivial first Ext.

7. Monoidal structure: intermediate results

In [ACD] the authors study monoidal structures on the homotopy category of representations up to homotopy. We follow the same path.

Theorem 9. Let $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ be homotopy characters. Then there exists a homotopy character $a \otimes b$, given by the formulas

$$(a \otimes b)_n = \sum_{i_1+\ldots+i_k=n} (a_{i_1} \otimes \ldots \otimes a_{i_k})(\Delta^{i_1-1} \otimes \ldots \otimes \Delta^{i_k-1})(b_n).$$

This tensor product of objects is strictly associative.

In the notation of [ACD], our formulas correspond to ω_0 and can be easily modified to give rise to any of ω_t.

In light of the results of [ACD], there is strong expectation that one can homotopy-consistently form tensor products of closed morphisms. There is little hope to extend this to non-closed morphisms and to have monoidal structure on the $\mathfrak{A}$ as opposed to $Ho(\mathfrak{A})$. Possibly, the situation can be remedied if all the formulas for homotopy limits are rewritten using a more suitable operad than A_∞ (namely, some Hopf operad).

References

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