NO REGULARIZATIONS OF PSEUDO-AUTOMORPHISMS WITH POSITIVE ENTROPY

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1. Introduction

From dynamical point of view the most interesting case of automorphisms of compact Kähler varieties is the case of automorphisms of positive entropy. The fundamental theorem of Gromov [Gro03] and Yomdin [Yom87] says that the topological entropy of the regular automorphism φ of a compact Kähler variety X is the following number:

$$h_{top}(\varphi) = \log \max_{1 \leqslant i \leqslant \dim(X)} \lambda_i(\varphi),$$

where λ_i is the *i-th dynamical degree of* φ. By definition it is the spectral radius of $\varphi^*|_{H^{i,i}(X)}$. The theory of automorphisms with positive entropy of compact Kähler surfaces is studied in details [Can99]. There are a lot of interesting examples of such automorphisms on K3 and rational surfaces.

Known automorphisms with positive entropy of surfaces induce a lot of examples of such automorphisms in higher dimensions. That is why we are mostly interested in those automorphisms that can not be induced in such a way. Thus, we study *primitive* automorphisms: those one where we have no dominant rational map $\alpha\colon X \to B$ and a rational automorphism $\varphi_B\colon B \to B$ with properties $0 < \dim(B) < \dim(X)$ and a diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{\varphi_B} & B \end{array}$$

However, examples of primitive automorphisms in dimensions 3 and higher are quite rare. In particular, there is just one known example of a rational threefold with a regular primitive automorphism of positive entropy [OT15]. Nevertheless, we can consider a wider class of automorphisms, namely *pseudo-automorphisms* those birational self-maps that do not contract divisors. The first dynamical degree is well-defined for such self-maps and if it is greater than 1 by [DS05] the entropy of such a map is positive. There are several examples of rational varieties with pseudo-automorphisms of positive entropy [Bla13], [PZ14].The reasonable question arises:

Question: if we have a primitive pseudo-automorphism φ with positive entropy of a smooth variety X then is there a smooth birational model of X on which this automorphism can be regularized?

Here we give some partial answer for this question and then show that the pseudo-automorphisms described in [Bla13] can not be regularized.

2. Obstruction to regularization of a pseudo-automorphism

We consider a smooth variety X and its pseudo-automorphism φ. We need an additional assumption on φ:

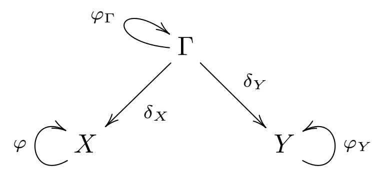
$$\lambda_1(\varphi)^2 > \lambda_2(\varphi). \tag{2.1}$$

Note that, if $\dim(X) = 3$ this assumption is true for φ or φ^{-1}.

Denote by θ_+ an eigenvector of the action of φ^* on $H^{1,1}_{\mathbb{R}}(X)$ corresponding to the eigenvalue λ_1. Then this is the criterion for the pseudo-automorphism φ that can be regularized.

Theorem 2.2. *Assume that $\varphi\colon X \to X$ is a primitive pseudo-automorphism with positive entropy and a property* (2.1). *If θ_+ is not nef, then there is no smooth birational models of X on which φ induces a regular automorphism.*

Proof. Assume that $\varphi_Y : Y \to Y$ is a regularization of φ on a smooth birational model Y of X. Then there exists a map $X \dashrightarrow Y$. Denote by Γ the graph of this map. By δ_X and δ_Y we denote the maps from Γ to X and Y respectively and by φ_Γ denote the birational automorphism of Γ induced by φ.



By [DS05, Corollary 7] we see that $\lambda_i(\varphi) = \lambda_i(\varphi_Y) = \lambda_i(\varphi_\Gamma)$. Moreover, since φ is a pseudo-automorphism and φ_Y is regular, the induced automorphism φ_Γ is also a pseudo-automorphism. In particular all these automorphism are 1-stable in sence of [Tru14]. Thus, by [Tru14, Theorem 1] in view of (2.1) all eigenvalues $\lambda_1(\varphi)$, $\lambda_1(\varphi_Y)$ and $\lambda_1(\varphi_\Gamma)$ are simple for maps φ^*, φ_Y^* and φ_Γ^* of $H^{1,1}(X)$, $H^{1,1}(Y)$ and $H^{1,1}(\Gamma)$. Denote by θ_{+Y} and $\theta_{+\Gamma}$ the eigenvectors corresponding to these eigenvalues. Since $\delta_X^*(\theta_{+X})$ and $\delta_Y^*(\theta_{+Y})$ are eigenvectors of φ_Γ^* of $H^{1,1}(\Gamma)$, we get the following:

$$\delta_X^*(\theta_{+X}) = \delta_Y^*(\theta_{+Y}) = \theta_{+\Gamma}\,.$$

Since $\theta_+ = \lim(\phi_Y^n)^* H$ for some ample class H and φ_Y is regular the divisor class θ_{+Y} is nef. Since the inverse image of any divisor D under birational morphism is nef if and only if D itself is nef, we have that classes coincide $\delta_X^*(\theta_{+X}) = \delta_Y^*(\theta_{+Y})$ and consequently θ_{+X} is nef. This contradicts to an assumption; thus, φ_Y is not regular. □

3. Blanc's pseudo-automorphism admits no regularizations

3.1. **Construction.** This family of pseudoautomorphisms with positive entropy is described in the papaer of Blanc [Bla13]. We consider a cubic hypersurface Q in $\mathbb{P}^3$. To each smooth point $p \in Q$ we associate a birational involution of the projective space

$$\sigma_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3.$$

The involution σ_p fix pointwise the hypersurface Q; its base locus contains of the point p and a curve $\Gamma \subset Q$.

Consider now k general distinct smooth points $p_1, \dots, p_k$ on Q and curves $\Gamma_1, \dots, \Gamma_k$ in the base loci of the involutions $\sigma_{p_1}, \dots, \sigma_{p_k}$. Consider a sequence of morphisms

$$\delta_i : X_i \to X_{i-1},$$

where X_{-1} is $\mathbb{P}^3$, then X_0 is the blow-up of all points $p_1, \dots, p_k$ and X_i is the blow-up of the strict transform of Γ_i. Denote by X the variety X_k and by δ the composition of all morphisms:

$$\delta : X \to \mathbb{P}^3.$$

Then by [Bla13, Theorem 1.2] the composition

$$\varphi = \sigma_1 \circ \cdots \circ \sigma_k : X \dashrightarrow X$$

is a pseudoautomorphism. Moreover, if $k > 2$, then the topological entropy of the composition is greater then zero.

Denote by H the class of hyperplane class in $\mathbb{P}^3$, by E_i the exceptional divisor over point p_i in X, by F_j the exceptional divisor over the Γ_j and by $\widetilde{H}$, $\widetilde{E}_i$ and $\widetilde{F}_i$ the inverse images of this divisors in X. These classes freely generate the Neron-Severi group of X.

Recall the necessary assertion by Blanc:

Lemma 3.1. [Bla13, Proposition 2.3] *If $\varphi = \sigma_1 \circ \cdots \circ \sigma_k$, then for all n there exists a set of non-negative numbers $\alpha_{n1}, \dots, \alpha_{nk}$ such that $\alpha_{ni} < \alpha_{n1}$ for all $i > 1$ and we have an equality:*

$$(\varphi^n)^*(\widetilde{H}) = \left(1 + 2\sum_{i=1}^{k} \alpha_{ni}\right)\widetilde{H} - \left(\sum_{i=1}^{k} 2\alpha_{ni}\widetilde{E}_i\right) - \left(\sum_{i=1}^{k} \alpha_{ni}\widetilde{F}_i\right).$$

3.2. **The map φ has no regularizations.** Now let us consider a general plane Π in $\mathbb{P}^3$ passing through the point p_1. Denote by C the curve of intersection of Π and Q and by $\widetilde{C}$ its strict preimage in X. Then $\widetilde{C}$ have the following intersection with divisors on X.

Lemma 3.2. *We have the following equalities:*

(i) $\widetilde{C} \cdot \widetilde{H} = 3$;

(ii) $\widetilde{C} \cdot \widetilde{F}_j = 6$ *for all* $j = 1, \ldots, k$.

(iii) $\widetilde{C} \cdot \widetilde{E}_1 = 1$ *and* $\widetilde{C} \cdot \widetilde{E}_i = 0$ *for* $i > 1$.

Denote as before by θ_+ the eigenvector of the action of φ^* on $\mathrm{NS}(X)$ which eigenvalue equals dynamical degree of φ.

Corollary 3.3. *If $k \geqslant 3$, then the class θ_+ is not numerically effective and φ can not be regularized.*

Proof. The product $\theta_+ \cdot \widetilde{C}$ is such a limit:

$$\theta_+ \cdot \widetilde{C} = \lim_{n\to\infty} \frac{(\varphi^n)^*(\widetilde{H}) \cdot \widetilde{C}}{\deg(\varphi^n)}.$$

Denote by A_n the sum $\sum_{i=1}^k \alpha_{ni}$. Then Lemmas 3.2 and 3.1 imply the following:

$$\frac{(\varphi^n)^*(\widetilde{H}) \cdot \widetilde{C}}{\deg(\varphi^n)} = \frac{(1+2A_n)\widetilde{H} \cdot \widetilde{C} - (\sum_{i=1}^k 2\alpha_{ni}\widetilde{E}_i \cdot \widetilde{C}) - (\sum_{i=1}^k \alpha_{ni}\widetilde{F}_i \cdot \widetilde{C})}{1+2A_n} =$$

$$= \frac{3(1+2A_n) - 2\alpha_{n1} - 6A_n}{1+2A_n} = \frac{3-2\alpha_{n1}}{1+2A_n}$$

Since by Lemma 3.1 we have $0 \leqslant \alpha_{ni} < \alpha_{n1}$ for all $i > 1$, then $\alpha_{n1} > \frac{A_n}{k}$. Thus, we get

$$\theta_+ \cdot \widetilde{C} \leqslant \lim_{n\to\infty} \frac{3 - \frac{2A_n}{k}}{1+2A_n} = -\frac{1}{k} < 0.$$

This proves that θ_+ is not nef. Using Theorem 2.2 we get the result. □

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